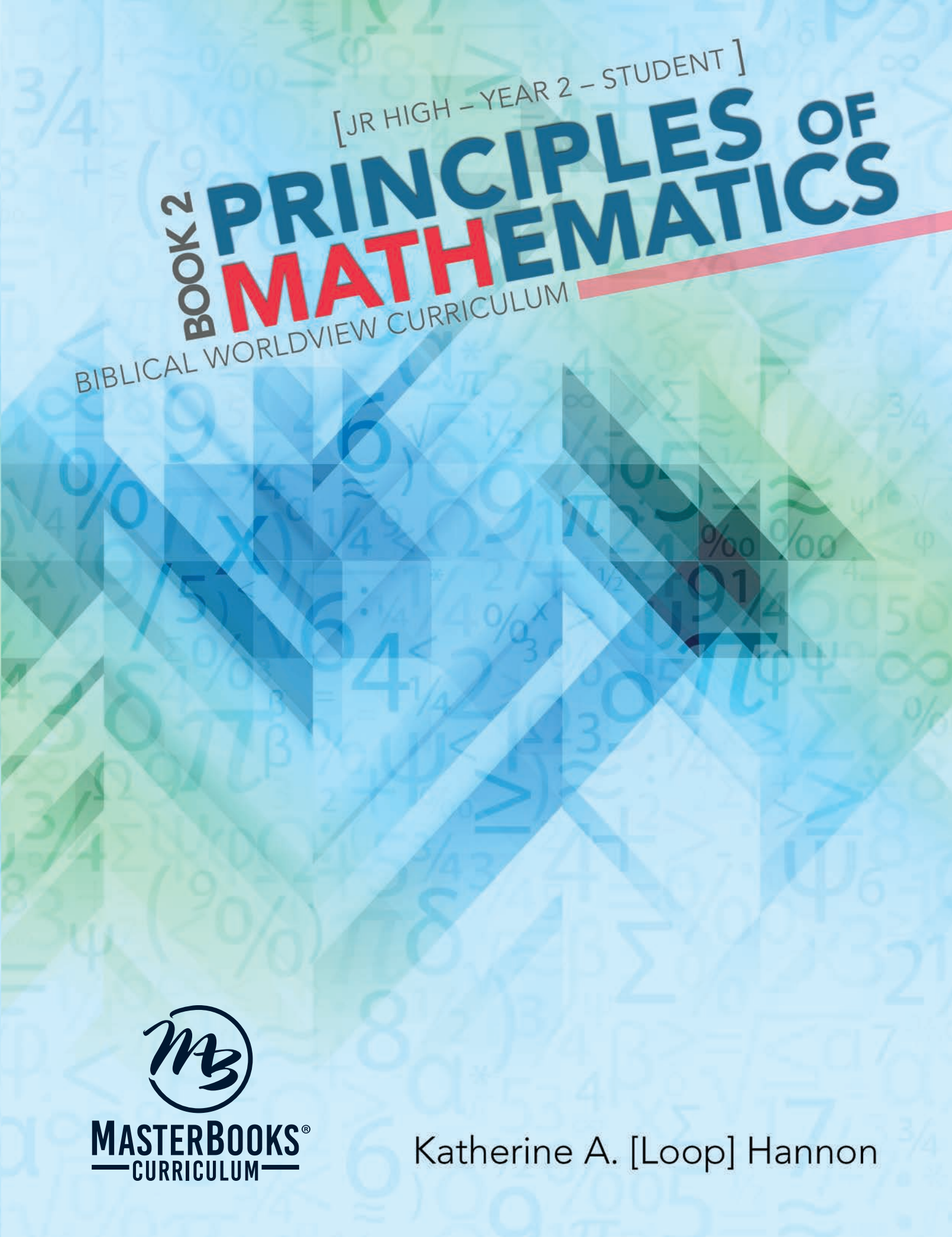
[JR HIGH – YEAR 2 – STUDENT]

BOOK 2

PRINCIPLES OF MATHEMATICS

BIBLICAL WORLDVIEW CURRICULUM



MASTERBOOKS®
CURRICULUM

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Acknowledgments

I don't believe you would be holding this curriculum in your hands today if it were not for some very special people. It's with much appreciation that I'd like to acknowledge and thank

my mom (Cris Loop) for being my right-hand helper throughout this project, solving hundreds of math problems and painstakingly working with me to make the material understandable.

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my brother Brian Loop for putting up with math questions at all hours of the day and helping me think through concepts and flow.

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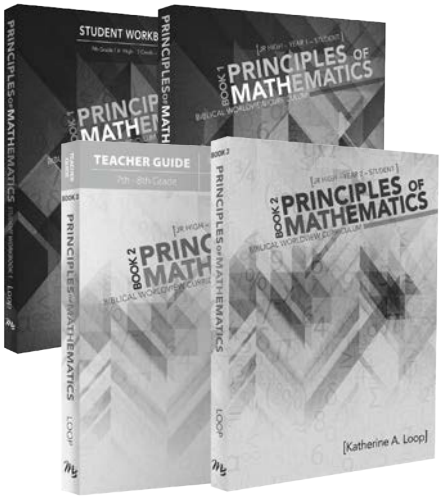
Above all, I'd like to thank the Lord, without whom all our labors are in vain.

Have You Completed Book 1?



The book you're holding in your hands builds on the foundational concepts covered in Book 1 of *Principles of Mathematics: Biblical Worldview Curriculum*.

Book 1 covers the core principles of arithmetic and geometry (along with some statistics), while Book 2 builds on those principles as it introduces the core principles of algebra, probability, and trigonometry (along with more statistics).

In Book 1, students learned to find the height of a tree without leaving the ground, use negative numbers to describe the force on objects, explore historical multiplication methods, apply math to music, and much more! In Book 2, students will build on that knowledge so they'll be prepared to understand and see the purpose of algebra and other upper math courses. From exploring genetics to force to sequences in music, they'll continue seeing how math truly is a real-life tool we can use to explore God's creation and serve Him.

About the Author

Katherine Loop is a homeschool graduate from northern Virginia. Understanding the biblical worldview in math made a tremendous difference in her life and started her on a journey of researching and sharing on the topic. For over a decade now, she's been researching, writing, and speaking on math, along with other topics. Her books on math and a biblical worldview have been used by various Christian colleges, homeschool groups, and individuals.

Website and Blog: www.ChristianPerspective.net
(Check out the free e-newsletter.)

About This Curriculum

This is Book 2 of a two-book math course designed to give students a firm mathematical foundation, both academically and spiritually. Not only does the curriculum build mathematical thinking and problem-solving skills, it also shows students how a biblical worldview affects our approach to math's various concepts. Students learn to see math, not as an academic exercise, but as a way of exploring and describing consistencies God created and sustains. The worldview is not just an addition to the curriculum, but is the starting point. Science, history, and real life are integrated throughout.

How Does a Biblical Worldview Apply to Math . . . and Why Does It Matter?

Please see Lesson 1.1 for a brief introduction to how a biblical worldview applies to math and why it matters.

Who Is This Curriculum For?

This curriculum is aimed at grades 6–8, fitting into most math approaches the year or two years prior to starting high school algebra. If following traditional grade levels, Book 1 should be completed in grade 6 or 7, and Book 2 (the book in your hands) in grade 7 or 8.

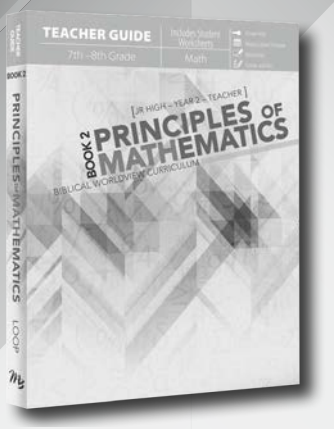
The curriculum also works well for high school students looking to firm up math's foundational concepts and grasp how a biblical worldview applies to math. High school students may want to follow the alternate accelerated schedule provided in the *Teacher Guide* and complete each year of the program in a semester, or use the material alongside a high school course.

Where Do I Go Upon Completion?

Upon completion of Book 1, students will be ready to move on to Book 2. Upon completion of both years, students should be prepared to begin or return to any high school algebra course.

Are There Any Prerequisites?

Book 1: Students should have a **basic knowledge of arithmetic** (basic arithmetic will be reviewed, but at a fast pace and while teaching problem-solving skills and a biblical worldview of math) and **sufficient mental development** to think through the concepts and examples given. Typically, anyone in 6th grade or higher should be prepared to begin.



Teacher Guide

Book 2: It is strongly recommended that students complete Book 1 before beginning Book 2, as math builds on itself, and the principles of arithmetic and geometry are essential for understanding the principles covered in Book 2. Before beginning Book 2, students need to be comfortable with basic math skills (including rounding and working with decimals), fractions, unit conversion, negative numbers, geometry formulas (finding perimeter, area, and volume), and exponents; they also need to have problem-solving skills.

What Are the Curriculum's Components?

The curriculum consists of the ***Student Textbook*** and the ***Teacher Guide***. The *Student Textbook* contains the lessons, and the *Teacher Guide* contains all the worksheets, quizzes, and tests, along with an Answer Key and suggested schedule.

How Do I Use This Curriculum?

General Structure — This curriculum is designed to be self-taught, so students should be able to read the material and complete assignments on their own, with a parent or teacher available for questions. If teaching in a classroom, the text can serve as the basis for the teacher's presentations. This *Student Textbook* is divided into chapters and then into lessons. The number system used to label the lessons expresses this order. The first lesson is labeled 1.1 because it is Chapter 1, Lesson 1.

Worksheets, Quizzes, and Tests — The accompanying *Teacher Guide* includes worksheets, quizzes, and tests to go along with the material in this book, along with a suggested schedule and answer key.

Answer Key — A complete Answer Key is located in the *Teacher Guide*.

Schedule — A suggested schedule for completing the material in one year, along with an accelerated one-semester schedule, is located in the *Teacher Guide*.

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Preface

In Book 1 of *Principles of Mathematics*, we explored together the core concepts of arithmetic and geometry, seeing how math's very existence declares God's praises, and how it serves as a real-life tool. While in that book we looked mainly *backwards* at aspects of math with which you were probably already familiar, in this course we'll be looking *ahead* at concepts into which you'll dig into more depth in future math courses. As we do, we'll see that all of these new concepts, like the ones we looked at in the last book, work because of God's faithfulness and are tools we can use to serve Him.

Researching for this second book has been a fun yet stretching process for me, as it has forced me to revisit many concepts that I saw as meaningless when I first learned them. The more I continued to learn and research, though, the more amazed I grew at how useful different concepts are in describing the intricacies of God's creation. I've joked that you know you're writing a math curriculum when you begin to "see" math everywhere...which for me was quite a switch.

You see, I didn't grow up loving math or seeing it as a way of describing God's creation. As I mentioned in the preface to Book 1, while I was good at math, I viewed it as a subject of rules to be memorized, applied, and forgotten. It wasn't until I read *Mathematics: Is God Silent?* by James D. Nickel as a senior in high school that I realized that math wasn't neutral or confined to a textbook—it was a way of describing God's creation that loudly proclaimed the Creator's praises.

Seeing the biblical worldview in math was transformational for me—so much so that I wanted to share it with everyone I could. It led to more than a decade of research on the topic, several books on teaching math from a biblical worldview, and lastly this curriculum series.

It's been such a joy to hear from users of Book 1 and my previous books about how they have blessed them. Hearing that one mom found her child reading extra lessons on her own, watching students' eyes light up at conventions when they realize math really is more than apparently meaningless bookwork, finding out that a biblical worldview of math encouraged a mom to overcome her fears of the

subject, and listening as a longtime teacher shared how excited she was to finally know how to teach math biblically has warmed my heart.

My earnest hope and prayer is that God will use this curriculum as well to bless readers. While the curriculum is labeled for junior high, it really applies to anyone of any age who wants to really "get" what math is all about from a biblical worldview. Whatever you're age and mathematical leaning (or lack thereof), I hope you'll find the material friendly and helpful. I hope you will find it a fascinating journey of discovery, helping you discover God's handiwork in math. Above all, I hope you'll leave encouraged that God can be trusted completely.

Writing this curriculum has been quite a journey for me, both in exploring math to a deeper level and of watching God faithfully provide. There were times I never thought the curriculum would actually reach completion, but it has. The acknowledgements don't begin to describe how blessed I was to be supported by a lot of talented people who were willing to let me hound them with questions, read countless drafts, work with me on refining and formatting concepts, check changes, and pray for and encourage me.

May the One who holds all creation together by the Word of His power (Hebrews 1:3) be honored and glorified in every page.

By God's Grace,

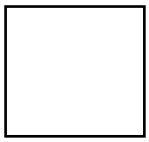
Katherine Loop

[CHAPTER 1]

The Big Picture

1.1 Overview of Mathematics

You probably learned to write numbers and count at a young age. At some point, you mastered the basic operations: addition, subtraction, multiplication, and division. Then came fractions, ratios, and percentages. Along the way, you learned to recognize different shapes, measure them, and find their perimeter, area, and volume.



86%

1 + 1 = 2

8 - 2 = 6

928.32

88 ÷ 2 = 44

5½ + 10½ = 16

We've become so familiar with math that it's easy to lose sight of math's purpose. What does 1 + 1 = 2 really mean? Why does 1 + 1 equal 2? Why do we have fractions, ratios, and percentages? What's the purpose of exploring shapes?

Before we jump into new concepts, we are going to pause and look at math's overall purpose — at the "big picture," so to speak. First, though, we need to talk about worldviews, as our worldview determines how we see both the "big picture" and the details of mathematics.

Worldview Matters

Whether we realize it or not, our perspective on any area of life is determined by what is called a worldview. In *Understanding the Times*, David Noebel (founder of Summit Ministries) defines a worldview this way: "A worldview is like a pair of glasses — it is something through which you view everything. And the fact is,

everyone has a worldview, a way he or she looks at the world."[1] In other words, a worldview is a set of truths (or falsehoods we believe to be true) through which we interpret life.

As we look at the "big picture" of math and study math together, we'll be seeking to start with the Bible — God's revealed Word to man — as our source of truth, using it as the "glasses" through which we look at math. We'll discover that the Bible gives us solid answers to where math originated, why math is possible, what we should expect as we use math, and how we should use math. With that in mind, let's begin looking at the "big picture" of math.

Where Did Math Originate?

Have you ever stopped to think about where math originated? How did we get math? Did men invent it? Has it just always been there?

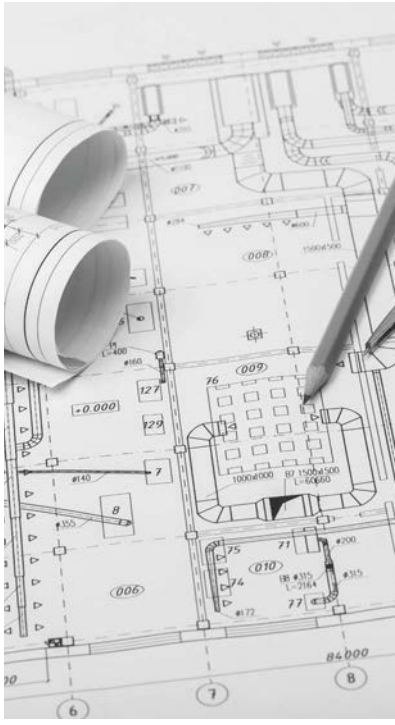
To understand *where* math came from, we have to first consider *what* math really is. Is math just a collection of facts in a textbook? While it may feel that way at times, math is hardly confined to a textbook. The main reason math is such an essential subject is because it applies outside of a textbook.

Take a moment to think about how many times you see numbers each day. Price tags, exit signs, clocks, calendars, addresses, expiration dates, nutritional data, Bible verses, speed-limit signs, rulers, radio dials, TV remotes, thermometers, store hours — we see numbers all the time.

Every occupation utilizes math to some degree or another. Scheduling airplane flights, generating computer graphics, managing businesses, designing bridges and buildings, laying out roads, producing digital sounds — all of these tasks require math. Even non-mathematical fields employ math. Artists deal with proportions, musicians work with a musical notation based on fractions, airplane pilots compute a weight and balance to make sure their plane will handle the cargo, veterinarians have to know how much medicine to give — math proves useful on the job, whatever that job might be.

Math is also used behind the scenes to develop or produce the technology and resources we use on a daily basis. Computers, cars, ovens, medical treatments, airplanes, cell phones, toys, GPS units — name something you use, and chances are math helped in developing or producing it. Pick a topic from science class — be it the distance to the stars or how an atom works — and most likely math has been used in its exploration.

All that to say, math is clearly more than a textbook exercise. **Math** is a way of describing the quantities and consistencies all around us — quantities and consistencies the Bible tells us the triune God created and sustains.



> *In the beginning God created the heaven and the earth. And the earth was without form, and void; and darkness was upon the face of the deep. And the Spirit of God moved upon the face of the waters (Genesis 1:1–2).*
>
> *In the beginning was the Word, and the Word was with God, and the Word was God. The same was in the beginning with God. All things were made by him; and without him was not any thing made that was made. . . . And the Word was made flesh, and dwelt among us, (and we beheld his glory, the glory as of the only begotten of the Father,) full of grace and truth (John 1:1–3, 14).*
>
> *Jesus answered them. . . . I and my Father are one (John 10:25, 30).*
>
> *For by him [Jesus] were all things created, that are in heaven, and that are in earth, visible and invisible, whether they be thrones, or dominions, or principalities, or powers: all things were created by him, and for him: And he is before all things, and by him all things consist (Colossians 1:16–17).*
>
> *[referring to Jesus] . . . upholding all things by the word of his power . . . (Hebrews 1:3).*

In other words, while men (using their God-given abilities) have developed the different symbols and techniques used in math over the years, the underlying principles those symbols and techniques describe are principles God created and sustains. In fact, as we'll see next, it's God's faithfulness that makes math possible.

Ignoring the Question

The fundamental question of why math works is one that has baffled secular philosophers for years, as it doesn't make sense apart from a biblical worldview. Consider this quote: "Some of the most abstruse concepts of mathematics have an uncanny way of becoming essential tools in physics. Many physicists have been so impressed by the usefulness of mathematics that they have attributed to it almost mystical power. . . . In this article I shall not attempt any deep philosophical discussion of the reasons why mathematics supplies so much power to physics. . . . The vast majority of working scientists, myself included, find comfort in the words of the French mathematician Henri Lebesgue: 'In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy.'"[2] In other words, this scientist is saying that he's going to avoid the fundamental question of why math works and just use it. As Christians, we don't have to ignore the question — the Bible gives us an answer.

Why Is Math Possible?

People who love math often say they love it because it's always predictable. You can count on 1 + 1 to equal 2. There's no subjectivity to math — it works the way it does.

Why is this? Again, the Bible gives us the answer. In Jeremiah 33:25, God tells us that He is keeping His covenant with the "ordinances" of heaven and earth.

> *Thus saith the LORD; If my covenant be not with day and night, and if I have not appointed the ordinances of heaven and earth;*

Day in and day out, our faithful, consistent Creator is holding all things together in such a predictable and reliable way that we can use math to describe it. Math is reliable *because* God is reliable. In fact, God uses the "ordinances" of heaven and earth to remind the Israelites that He'll be that reliable to His covenant with them.

> *Then will I cast away the seed of Jacob and David my servant, so that I will not take any of his seed to be rulers over the seed of Abraham, Isaac, and Jacob: for I will cause their captivity to return, and have mercy on them (Jeremiah 33:26).*

Water Droplets and Addition

It's important to note that when we say the universe is consistent, we don't mean that it all operates exactly the same way. For example, one water droplet plus another water droplet does *not* equal two water droplets — instead, they merge to form one larger droplet. The apparent contradiction in how liquids combine reminds us that, no matter how well we think we have things figured out, God's laws and universe are more complex than we can imagine. God has different, though equally consistent, principles for governing liquids at the visible level than He does solids.

Note, though, that while we visibly see one larger droplet instead of two, that is not what occurs on the atomic level. At the atomic level, the starting atoms plus the final atoms add together following the rules of regular addition — if we started with 1 atom and added 1 more atom, the ending larger droplet would have 2 atoms.

Men have often sought to make either experience or man's intellect the foundation for the underlying truth in math. Yet neither of these foundations hold up to scrutiny. Raindrops baffle those who would say truth in math is determined solely intellectually,[3] as it's experience that tells us one raindrop plus another equals a larger one. Yet experience can't be the source of truth either, as we use concepts in math we've never experienced (such as infinite numbers). In a biblical worldview, though, we have a solid answer for why we're able to learn both experientially and intellectually — God both created the world and gave us the ability to explore it.

The Bible also gives us a reason for understanding why we're able to discern and record the orderly world God created. It tells us God created man in His image and gave man dominion over the earth.

> *And God said, Let us make man in our image, after our likeness: and let them have dominion over the fish of the sea, and over the fowl of the air, and over the cattle, and over all the earth, and over every creeping thing that creepeth upon the earth. So God created man in his own image, in the image of God created he him; male and female created he them (Genesis 1:26–27).*

Thus we're capable of exploring the universe and developing different symbols and techniques to describe it.

What Should We Expect as We Use Math?

Not only does the Bible give us principles that explain where math came from and why it works, but it also helps us know what to expect as we use math to describe the quantities and consistencies around us. The Bible tells us that God created a perfect world, but sin marred that world.

> *And God saw every thing that he had made, and, behold, it was very good. . . . (Genesis 1:31)*

> *Wherefore, as by one man sin entered into the world, and death by sin; and so death passed upon all men, for that all have sinned: (Romans 5:12).*

Thus we should expect to find evidence of design (pointing us to the wisdom and care of the Creator), yet at the same time a world marred from that perfect order (reminding us of the Fall). And guess what? That is exactly what we find.

For a simple example, we can observe that cats have four legs. Math helps us describe this aspect of the design God placed in cats. Yet if you were to look at enough cats, you'd eventually find a cat that was missing a leg, as we live in a fallen world.

How Should We Use Math?

The Bible tells us to do all that we do in the name of Jesus and with thanksgiving to God.

> *And whatsoever ye do in word or deed, do all in the name of the Lord Jesus, giving thanks to God and the Father by him (Colossians 3:17).*

Notice that this verse doesn't say, "and whatsoever ye do *but math.*" It says "and whatsoever ye do." Math is not exempted. Within math, as in every other area of life, we have an opportunity to depend on and praise God. We can also use math, along with other skills we learn, to help us in the tasks God has given us to do.

Term Time
Here are a few terms we'll use extensively as we explore the principles of mathematics together. Be sure to familiarize yourself with them before we get started.
Expression — An expression is "a collection of symbols that jointly express a quantity."[4] For example, 4 + 5 is an expression — 4, 5, and + are a collection of symbols that together express the quantity 9.
Equation — An equation is "a statement that the values of two mathematical expressions are equal (indicated by the sign =)."[5]
Simplify — When we refer to simplifying an expression or equation, we mean to express it as simply as possible. For example, 5 + 6 simplifies to 11.

Keeping Perspective

Wow — that's quite an amazing "big picture," isn't it? Think about it for a minute. Every time you use math and see that it works it's reminding you that you can trust God. Math's an exploration of God's handiwork — a tool we can use to explore His creation while worshiping the Creator.

I hope you're excited about digging deeper into math this year. As we do, we'll see over and over again how math helps us describe the quantities and consistencies God created and sustains. We'll learn how to use concepts outside a textbook, applying what we learn to help us with the tasks God's given us to do. Above all, we'll reflect on what a faithful, wise, infinite Creator we serve — a Creator worthy of all our trust.

1.2 The Language of Mathematics — Symbols and Conventions

Before we jump into the details of math, we're going to take the rest of this chapter to continue building the "big picture." In this lesson, we're going to look at the language side of mathematics.

Many aspects of math can be thought of as a language system. Mathematical symbols, notations, and conventions are like a language system to communicate about the quantities and consistencies God created and sustains.

Symbols

For years, you've been using symbols to **name** quantities. For example, you might use "5" to represent five cups of flour, five inches, or five additional CDs. You've also used symbols to **identify** (the numbers in a telephone number identify a specific telephone line) and to **order** (such as the 5th most popular song of the year).

You've also used symbols such as – (minus), + (plus), x (times), ÷ (division), = (equals), and ≠ (does not equal) for years. These symbols make it simpler to write and solve equations. (After all, can you imagine having to write "plus" every time you want to show the addition of two numbers instead of writing "+"?)

As we did in Book 1, we'll be representing multiplication different ways. All four methods shown here are legitimate ways of representing 4 times 5.

4 x 5 4 • 5 4(5) (4)5

Note: A number written next to a set of parentheses means to multiply that number by whatever is inside the parentheses. For example, 4(5) means to multiply 5 by 4.

It's important to understand that the symbols with which we're familiar are but *one way* of describing God's creation. There are others! The chart shows some of the different ways a single quantity has been represented. Again, the symbols we use are merely part of a language system to describe the quantities and consistencies God created and sustains.

Different Symbols That Have Been Used to Express a Single Quantity[5]

 Early Arab	 Mayan/Aztec	 Hebrew (Rabbinical)	 Modern
 Hebrew	 Samaritan	 Babylonian	 Greek
 Egyptian (Hieroglyphic)	 Roman	 Early Sumerian	 Egyptian (Hieratic Variant)
 Old Chinese	 Old Japanese	 Cretan	 Greek (Herodianic)

Math, Adam, and the Animals

Much as Adam used names back in the Garden of Eden to describe the animals, we use names like "three" and symbols like "3" to describe various aspects of God's creation. Much as Adam named the animals while in the Creator's presence, through Jesus we can again use math while fellowshipping with the Creator.

Notations

Place value, fractions, decimals, percents, exponents, etc. — these are all notations (a "series or system of written symbols used to represent numbers, amounts, or elements . . ."[7]) that help us represent amounts. Since we covered all of these notations thoroughly in Book 1, they all should already be familiar to you. We'll encounter them again as we explore math together.

It's worth a quick reminder, though, that our decimal system (the notation we typically use to represent quantities) is a place-value system (a system in which the place, or location, of a symbol determines its value) based on 10. Each place to the left is worth 10 times the previous place, and each place to the right is worth $\frac{1}{10}$ of the previous place. The decimal point separates the whole numbers from the partial ones.

Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths
	,			.			
1,000 or $(10 \bullet 100)$	100 or $(10 \bullet 10)$	10 or $(10 \bullet 1)$	1 or $(10 \bullet \frac{1}{10})$		$\frac{1}{10}$ or $(10 \bullet \frac{1}{100})$	$\frac{1}{100}$ or $(10 \bullet \frac{1}{1{,}000})$	$\frac{1}{1{,}000}$ or $(10 \bullet \frac{1}{10{,}000})$

The rules you've learned for performing operations on whole and decimal numbers help keep track of place value automatically.

Conventions

Conventions are agreed-upon protocols or rules that aid us in communication.

All conventions apply within a context. Some conventions may only be followed for a specific situation or industry. Other conventions are followed throughout our typical number system.

One important convention that applies throughout our typical number system is the order of operations. The **order of operations** is an agreed-upon order in which to read mathematical symbols and notations.

While the order of operations applies throughout our typical number system, it's possible to follow a different convention. In fact, some calculators and computer programming languages have/do. Every convention applies within a context!

The Order of Operations

1) Solve anything within parentheses first, using the following order (which is the same order we'll follow outside of parentheses too):
   a) Simplify exponents and roots, from left to right.*
   b) Multiply or divide, from left to right.
   c) Add or subtract, from left to right.
2) Next, simplify exponents and roots, from left to right.*
3) Now multiply or divide, from left to right.
4) Last, add or subtract, from left to right.

**There will not be any exponents or roots in your problems until they are reviewed in Lesson 1.5 (exponents) and Lesson 17.1 (roots).*

The order of operations helps us know what's meant by an expression. For example, consider $4 + 12 \div 2$. What does this equal? Do we need to divide 12 by 2, and then add 4, in which case we would get 10, or do we need to add the 4 and 12 and then divide the sum (16) by 2, in which case we would get 8?

The order of operations makes it clear the order in which we should complete the operations. We divide first, and then add.

$$4 + 12 \div 2$$
$$4 + \quad 6 = 10$$

But if there had been parentheses involved, we would have solved whatever was inside the parentheses first.

$$(4 + 12) \div 2$$
$$16 \quad \div 2 = 8$$

Did you know?
Years ago, other methods, such as a line above or below numbers, were used to indicate groupings instead of parentheses. An example is shown below.

$\overline{4 + 5} \bullet 3$
instead of
$(4 + 5) \bullet 3$

Parentheses are just a convention used to show groupings.

Parentheses and Assumed Groupings

Even though the first point in the order of operations is listed as "solve anything within parentheses first," the principle is to **solve groupings first**.

If we saw $2\overline{)5 + 6 + 9}$, we'd know that we need to add the numbers underneath the division sign before completing the division. Although there aren't parentheses, it's a clearly understood grouping. The *sum* of $5 + 6 + 9$ is the number we need to divide. The first point of the order of operations is to deal with grouped numbers first. Parentheses are just a common convention used to show groupings.

Keeping Perspective

Much of studying math is similar to learning a language — we learn the symbols, notations, and conventions we've adopted to help us describe God's creation. While it's easy to get lost in the language side of math, remember that the purpose of a language is to communicate . . . and the purpose of the many symbols, notations, and conventions you'll encounter in math is to communicate about the quantities and consistencies God created and sustains.

1.3 The Consistency of Mathematics — Operations and Properties

Math symbols, notations, and conventions are useful because they help us describe the underlying consistencies all around us — consistencies God created and sustains. Let's do a quick "big picture" review of the consistency of mathematics.

Operations and Properties

1 + 1 consistently equals 2. Why? Because God holds all things together in a consistent, predictable fashion. As we read in Lesson 1.1, God keeps His covenant with the "ordinances" of heaven and earth.

> *Thus saith the LORD; If my covenant be not with day and night, and if I have not appointed the ordinances of heaven and earth; Then will I cast away the seed of Jacob and David my servant, so that I will not take any of his seed to be rulers over the seed of Abraham, Isaac, and Jacob: for I will cause their captivity to return, and have mercy on them (Jeremiah 33:25–26).*

God's faithfulness enables us to memorize addition, subtraction, multiplication, and division facts, as well as to use **algorithms**, or step-by-step methods, to solve problems we don't have memorized. Algorithms work because of the underlying consistency God created and sustains. While there are many different algorithms we could use to add, subtract, multiply, and divide, the ones traditionally taught are shown below. We saw in Book 1 how each of these algorithms helps us break down problems we don't have memorized into problems we do, keeping track of place value for us.

Traditional Addition, Subtraction, Multiplication, and Division Algorithms

$$\begin{array}{r} {\scriptstyle 1} \\ 12 \\ +19 \\ \hline 31 \end{array} \qquad \begin{array}{r} {\scriptstyle 1\ 1} \\ 2\not{4} \\ -\ 9 \\ \hline 15 \end{array} \qquad \begin{array}{r} {\scriptstyle \not{1}} \\ 12 \\ \times 29 \\ \hline 108 \\ +240 \\ \hline 348 \end{array} \qquad \begin{array}{r} 24 \\ \hline 2\overline{)48} \\ -\ 4 \\ \hline 08 \\ -\ 8 \\ \hline 0 \end{array}$$

God's faithfulness also allows us to make observations about the properties of operations and apply those properties to other situations. A **property** is a characteristic that holds true in every situation. Basically, a property is a way of expressing a truth about the ordinances God created and sustains. Take a moment to review these properties we covered in Book 1, as we'll be applying them over and over again in this book too.

Commutative Property of Addition and Multiplication

Order doesn't matter.

Addition	Multiplication
$1 + 2 = 2 + 1$	$2 \cdot 3 = 3 \cdot 2$
$3 = 3$	$6 = 6$

Associative Property of Addition and Multiplication

Grouping doesn't matter.

Addition	Multiplication
$(1+2) + 3 = 1 + (2+3)$	$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$
$3 + 3 = 1 + 5$	$6 \cdot 4 = 2 \cdot 12$
$6 = 6$	$24 = 24$

Identity Property of Multiplication

Multiplying by 1 doesn't change the value.

$$2 \cdot 1 = 2$$

Note: It follows that dividing by 1 doesn't change the value of a number either,[8] *as division simply reverses multiplication. 2 • 1 equals 2, and 2 ÷ 1 equals 2.*

Test the properties we're reviewing today with any objects you have around the house. For example, add quantities with different groupings — the grouping won't matter. Again, these properties describe what happens in real life. Properties are a way of expressing truths about the ordinances God created and sustains.

Identity Property of Addition

Adding 0 doesn't change the value.

$$2 + 0 = 2$$

The Distributive Property of Multiplication

If you distribute multiplication among addends and then add those results together, you'll get the same result as if you first did the addition and then multiplied by the sum.

$2(3 + 4)$	$2(3 + 4)$
	$2(3) + 2(4)$
$2(7) = 14$	$6 + 8 = 14$
14 total	**14 total**

Note: It doesn't matter which side of the parentheses the multiplier is on. The following would also hold true:

(3 + 4)2	(3 + 4)2
	(3)2 + (4)2
(7)2 = 14	6 + 8 = 14
14 total	**14 total**

Worldview Matters

Take a look at this typical property presentation:

> *A property describes the way something is. We can't change properties. We are stuck with properties because they are what they are.*[9]

Let's reword this to convey a biblical worldview instead.

> *A property describes the way God has chosen to govern an aspect of His creation. We can't change properties because we can't change God. We are stuck with properties because God is who He is and He is going to govern creation the way He has decided to govern creation whether or not it's popular with us.*

Do you see the different message conveyed? One leaves us looking at properties . . . the other lifts our eyes to the One holding properties together.

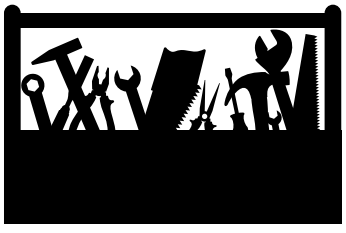
As you study math or any other subject, filter what you read through a biblical worldview, giving God the glory for His creation.

Keeping Perspective

Every time you solve a math problem, you're relying on the underlying consistency present in math. And every time you see that math still operates consistently, it's testifying that God is still on His throne, faithfully holding all things together.

For ever, O LORD, thy word is settled in heaven (Psalm 119:89).

1.4 The Usefulness of Mathematics — Problem Solving



Since we live in a consistent universe held together by a consistent, faithful God, and since math helps us describe those consistencies, we should expect math to prove quite useful outside of a textbook. In fact, Walter W. Sawyer compares math to a chest of tools.

> *"Mathematics is like a chest of tools: Before studing the tools in detail, a good workman should know the object of each, when it is used, how it is used, what it is used for." — Walter W. Sawyer*[10]

> *Before studying the tools in details, a good workman should know the object of each, when it is used, how it is used, what it is used for.*[11]

I love that analogy. Tools come in all sorts of shapes and sizes. Some are intuitive; others require practice to use effectively. Each one, though, serves a useful purpose. As we study math together, we're going to discover how the various aspects of math are like tools. Each one, serves a useful purpose. And as with actual tools, sometimes it takes multiple mathematical tools to get a job done.

Problem-Solving Steps

In order to help you learn to see and use math as a practical tool, we'll be including a good deal of problem-solving practice throughout this course. These problems will give you practice in applying the tools you learn. When you encounter a word problem (or a real-life problem), here are some helpful steps to follow:

1. Define — Write out the information you know and the information you need to find.

2. Plan — Think through how you can find the information you need to find — it may take more than one step. Here's where you need to look at the information you know and see how it relates to the information you need to find. Can you find the information you're seeking by addition? Subtraction? Multiplication? Division? A combination? It's often helpful to use words or symbols to write out equations showing how the information you know relates to the information you're trying to find.

3. Solve — Here's where you do the math.

4. Check — Does the answer make sense?

This example was taken from an early 1900s math book[11]—prices have gone up since then!

Example: The average yield of a certain field is 36 bushels of corn per acre. By putting in an average of 2 days per acre more time cultivating the crop, the farmer is able to increase the yield to 54 bushels per acre. If corn is worth 45 cents per bushel, how much per day does he earn for his extra work?

1. Define — Let's look at the information we've been given.

original average yield = 36 bu

increased average yield = 54 bu

increase in time = 2 days

sales price = $0.45

earnings per additional day = ?

2. Plan — We want to find the earnings per additional day. These earnings will be the increase in total dollars the farmer receives for the additional corn (i.e., the increase in earnings) divided by the increased time.

earnings per additional day = increase in earnings ÷ increase in time

We know the increase in time is 2 days. But how much more do we make for that increased time? That is, what's our increase in earnings? To find that, we have to first figure out how many more bushels we're getting. Multiplying the increase in bushels by the sales price at which we sell those bushels will tell us the increase in earnings.

increase in number of bushels = original average yield – increased average yield

increase in earnings = increase in number of bushels • sales price

Notice how it will take a few steps, but we can find the information we need to find from the information we were given.

3. Solve — Let's use the values we've been given to execute our plan.

increase in number bushels = 54 bu – 36 bu = 18 bu

increase in earnings = 18 • \$0.45 = \$8.10

earnings per additional day = \$8.10 ÷ 2 = \$4.05

4. Check — \$4.05 seems to be a reasonable answer considering that we have 18 bushels more to sell, each one sells for about a half a dollar, and it took 2 extra days worth of work. (We rounded and mentally estimated to see that \$4.05 seemed reasonable.) If we were uncertain of our math or got an answer that seemed unreasonable, we could go back and check our math using the reverse operation too.

Checking our subtraction with addition:

18 bu + 36 bu = 54 bu

Checking our multiplication with division:

\$8.10 ÷ 18 = \$0.45

Checking our division with multiplication:

\$4.05 • 2 = \$8.10

You don't necessarily need to write out every step in the word problem — they're listed here to walk you through the process. You'll often be able to define, plan, and check mentally, only writing down the actual solving step. However, it's important to be familiar with the process of

defining (thinking about what you know),

planning (figuring out how to find what you need to know),

solving (doing the math), and

c**hecking** (making sure your answer makes sense).

It's also important to note that **you can use letters to stand for the different numbers in a word problem rather than words.** For example, in the example, we could have written *b* instead of "increase in number of bushels," *x* instead of "increase in earnings," and so forth.

Keeping Perspective

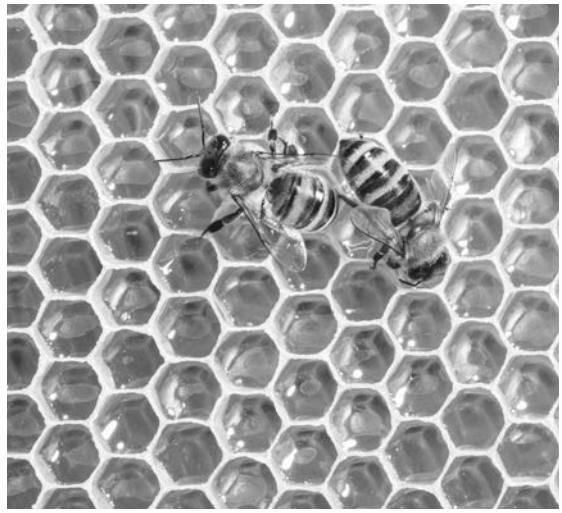
As we continue looking at the principles of mathematics together, we're going to look at how the concepts we learn serve useful purposes. We'll also include opportunities for you to learn to use those tools yourself in a useful way — after all, a tool doesn't do much good if you don't know how to use it. Since math applies in every occupation, knowing how to use it can help you with whatever tasks God assigns you, whether at home or on the job.

1.5 Overview of Geometry

As we saw in Book 1, math helps us describe and explore the shapes all around us. Without going into too much detail, let's take a "big picture" review of the branch of math known as geometry.

Understanding Geometry

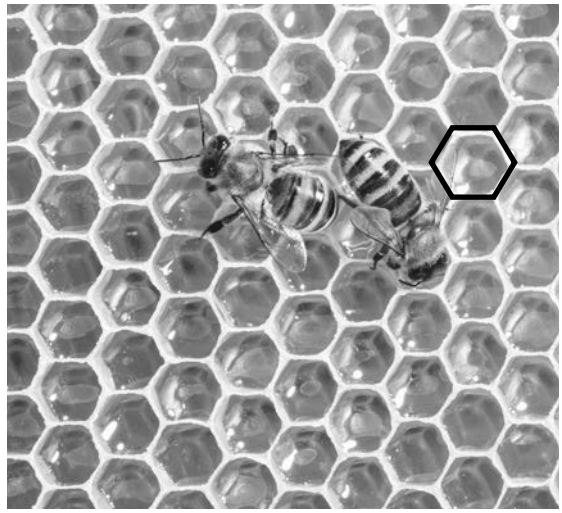
Geo means "earth" and *metron* means "measure," so *geometry* actually means "earth measure." Both directly and indirectly, geometry helps us "measure the earth."


Naming Shapes

Shape names give us a handy way to refer to shapes with specific properties. While we won't review all the shape names we covered back in Book 1 here, you'll review some of them on Worksheet 1.5. (All worksheets are located in the *Teacher Guide*.)


rectangle


hexagon


triangle

Measuring Shapes

We often find ourselves measuring different aspects of shapes. Let's do a quick review of units of measure, perimeter, area, and volume, all of which we looked at thoroughly in Book 1.

First, though, let's talk for a minute about formulas. We use formulas to help us find perimeter, area, and volume. As we discussed in Book 1, a **formula** is "a mathematical relationship or rule expressed in symbols."[12] For example, the area of a rectangle always equals the length times the width. We can express this as a formula like this:

$$A = \ell \cdot w$$

Here we've used the symbols A, ℓ, and w to stand for the area, length, and width of a rectangle.

It's worth noting that formulas rely on consistency. If real-life shapes constantly morphed from one shape into another like Silly Putty, we wouldn't be able to represent mathematically how the area related to the length and width (at least not with it meaning anything outside of a textbook). Once again, wherever we use math, we find it pointing us to our faithful, consistent Creator and Sustainer.

Formulas describe all sorts of different mathematical relationships, not just those within geometry. For example, if you make an hourly wage, then the mathematical relationship between the hours you work and the money you make a day at that job could be written as a formula.

$$\textit{money made per day} = \textit{hourly wage} \cdot \textit{number of hours worked}$$
$$m = w \cdot h$$

Be sure to take some time to review the geometry formulas we covered in Book 1 (you'll find them in the appendix at the back of this book, as well as in the reference sheets you were given from the *Teacher Guide).*

Units of Measure

To measure, we need a standard against which to measure. There are many different **units of measure** we could use to measure (inches, feet, meters, cubic feet, gallons, etc.). We use abbreviations to express units (in, ft, m, ft^3, gal, etc.).

Remember to always include the unit of measure (which we'll often refer to as simply the **unit**) in problems where one is given. For example, if a problem has you add 2 in and 7 in, give your answer as 9 in, not 9.

Perimeter

The **perimeter** is the distance around a closed shape. Knowing how to find the perimeter of various shapes helps us figure out how much edging to buy for a garden (or crown molding for a room), the fencing needed to enclose a yard, the distance around a basketball court, etc.

Example: Find the perimeter of the raised rectangular garden bed shown if its length is 8 ft and its width is 3.25 ft.

The perimeter of a rectangle equals 2 times the length plus 2 times the width.

$P = 2 \bullet \ell + 2 \bullet w$

Knowing this, we only need to substitute the values for length and width to find the perimeter.

$P = 2 \bullet \mathbf{8\ ft} + 2 \bullet \mathbf{3.25\ ft}$

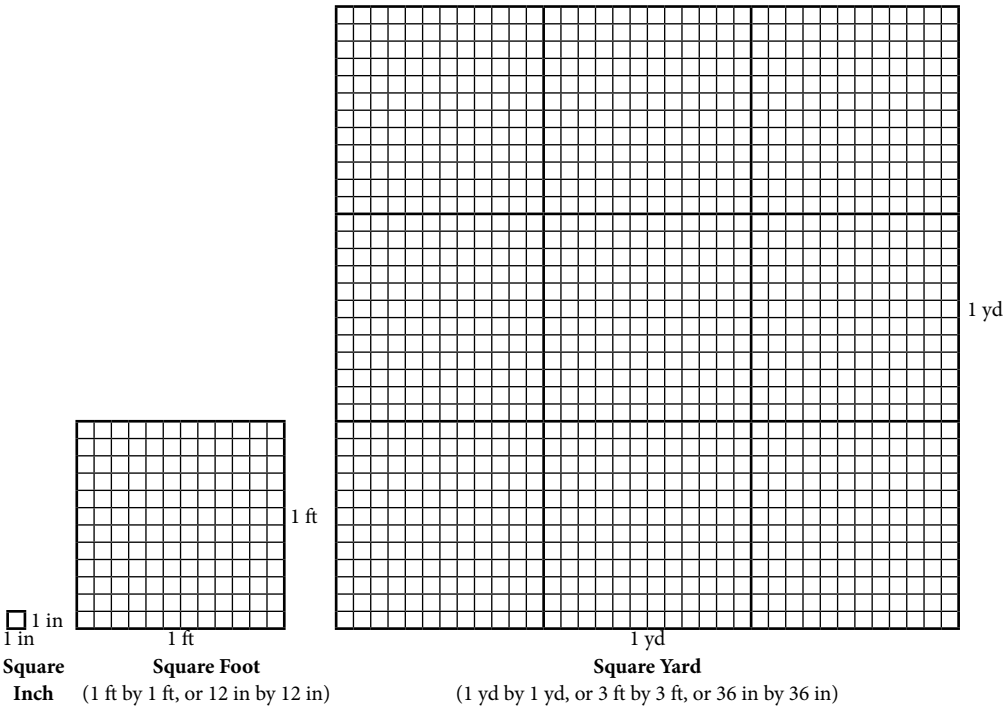
$P = \mathbf{16\ ft} + \mathbf{6.5\ ft}$

$P = \mathbf{22.5\ ft}$

Area

Area is the space a two-dimensional shape encloses. Finding the area helps us find the carpet needed to cover the floor of a room, the paint needed to paint the walls of a room, the amount of grass needed to cover a yard, etc.

We measure area in square units — that is, units that are 1 unit wide by 1 unit long. In other words, a square inch is a square with 1-inch sides, a square foot is a square with 1-foot sides, and a square yard is a yard with 1-yard sides.



Square Inch

Square Foot (1 ft by 1 ft, or 12 in by 12 in)

Square Yard (1 yd by 1 yd, or 3 ft by 3 ft, or 36 in by 36 in)

We typically use exponents to represent squared units. For example, an area of 60 square feet would be written like this: 60 ft^2. The superscript ^{2} is an **exponent** — a shorthand way of representing repeated multiplication. In 60 ft^2, the ft^2 means that ft is multiplied by itself (ft^2 = ft • ft).

Example: Find the area of the raised rectangular garden bed shown if its length is 8 ft and its width is 3.25 ft.

The area of a rectangle equals the length times the width.

$A = l \bullet w$

Knowing this, we've only to substitute the values for length and width to find the area.

$A =$ **8 ft • 3.25 ft**

$A =$ **26 ft^2**

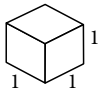
Notice that the area is in ft^2. If we do the math correctly, area will be in square units. Here, we multiplied 8 ft by 3.25 ft . . . which gave us 26 ft • ft . . . which we wrote as 26 ft^2.

Volume

Volume is the term used to describe the space inside a *three-dimensional*, or *solid*, object. Finding the volume helps us figure out such real-life problems as how much space a shipping container encloses, how much dirt a raised garden requires, and how much water a pool can hold.

We measure volume in cubic units — that is, units that are 1 unit wide by 1 unit long by 1 unit high. In other words, a *cubic* inch is a *cube* with a width, height, and length of 1 inch.

1 Cubic Unit



Exponents can be used with numbers as well as units. 6^2 is a shorthand way of writing 6 • 6 — that is, 6 multiplied by itself. For a review of exponents, look back at Lesson 16.1 in Book 1, or ahead at Lesson 15.1 in this book.

As with squared units, we typically use exponents to represent cubic units. For example, a volume of 60 cubic feet would typically be written like this: 60 ft^3. The superscript ^{3} next to ft means that ft is multiplied by itself three times (ft^3 = ft • ft • ft). Notice from the following example that when finding the volume, we do end up multiplying ft by ft by ft.

Example: Find the volume of the raised rectangular garden bed shown if its length is 8 ft and its width is 3.25 ft and its height is 1 ft.

We know the volume of a rectangular prism equals the area of the base times the height.

$V = B \bullet h$

And since the base is a rectangle, we can find the area of our base by multiplying the length times the width. Knowing this, we only need to substitute the values to find the volume.

$V =$ **8 ft • 3.25 ft • 1 ft**

$V =$ **26 ft^3**

Keeping Perspective

Since geometry helps us measure the shapes all around us, it's a very useful application of math. While we won't be exploring geometry as much in this course as we did in Book 1, we will add a few additional tools to our "measure shapes" toolbox, as well as discover some ways geometric "tools" help us explore other aspects of God's creation too.

1.6 Chapter Synopsis

I hope our "big picture" look at math has you excited about studying it this year. Usually the chapter synopsis will include a review of the main points or some tidbit from history or science. However, since this entire chapter was review, I'll simply end with a reminder that math is an exciting exploration of God's handiwork and a useful tool we can use to serve Him.

[CHAPTER 9]

Parentheses, the Distributive Property, and Unknowns

9.1 Parentheses and Nested Parentheses

Back in Lesson 1.2, we reviewed the order of operations, an agreed-upon convention that tells us the order in which the operations in an expression should be completed when simplifying the expression. This order tells us to solve parentheses, or grouped numbers, first, then exponents and roots, then multiplication and division, and then addition and subtraction. By having an agreed-upon order, it's clear what an expression means.

Example: Simplify $\frac{5}{9}(20 - 32)$

Solve within the parentheses first:

$\frac{5}{9}\mathbf{(-12)}$

Multiply $\frac{5}{9}$ by –12 and simplify:

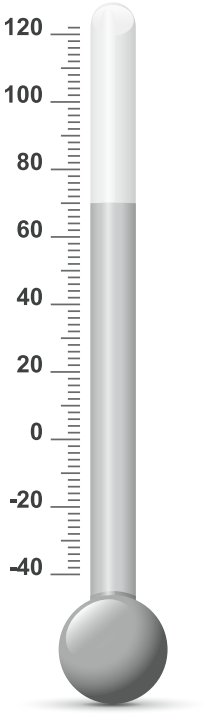
$-\frac{\mathbf{60}}{\mathbf{9}}$

$\mathbf{-\frac{20}{3}}$

$\mathbf{-6\frac{2}{3}}$

We're going to take this chapter to take a closer look at problems containing parentheses. Along the way, we'll review the distributive property and apply it to help us solve problems containing both parentheses and an unknown.

Let's start our explorations by looking at how parentheses can help us describe converting temperature from Fahrenheit to Celsius.



Parentheses and Temperature Conversions

The conversion between Fahrenheit and Celsius isn't as straightforward as other conversions. To convert Fahrenheit to Celsius, we have to subtract 32 degrees and then multiply that result by $\frac{5}{9}$.

Example: Convert 70 degrees Fahrenheit to degrees Celsius.

Subtract 32°:

$70° - 32° = 38°$

Multiply the result by $\frac{5}{9}$:

$38° \bullet \frac{5}{9} = \frac{190°}{9} = 21.11°$

70 degrees Fahrenheit is 21.11 degrees Celsius.

Now, we could have used parentheses to express this problem more concisely.

We can represent that we need to subtract 32° from 70° and *then* multiply the difference by $\frac{5}{9}$ like this:

$$\frac{5}{9}(70° - 32°)$$

Notice that we used the parentheses to group together the operation that needed to be completed first, thereby writing the entire process in one expression rather than as two separate expressions.

Example: Convert 70 degrees Fahrenheit to degrees Celsius.

$\frac{5}{9}(70° - 32°)$

Solve within the parentheses:

$\frac{5}{9}(\mathbf{38°})$

Complete the multiplication:

$\mathbf{\frac{190}{9}}$

Convert to a decimal:

21.11°

In fact, we could go one step further and use parentheses to help us write a formula for converting between Fahrenheit and Celsius.

Formula for Converting Between Fahrenheit and Celsius:

$$\boldsymbol{C = \frac{5}{9}(F - 32°)}$$

where C = degrees in Celsius and F = degrees in Fahrenheit

Nested Parentheses

So far, we've limited our use of parentheses to a single pair of parentheses. But as you continue to study God's creation in greater detail, you'll find yourself needing additional groupings at times. For example, suppose that, per person, it costs \$2 (shoe rental) plus \$4 (bowling cost) to bowl. If there are 5 people in a family and the family wants to budget enough money for all 5 people to bowl 7 times a year, how much should they budget?

Here, we need to add the \$2 and \$4 to find the cost per person, *then* multiply that by 5 people to find the cost for the entire family each time they bowl, and *then* multiply that by the number of times they want to budget for per year (7).

Rather than writing one expression for the addition, another for the first multiplication, and another for the final multiplication, we can show the entire problem as one expression by using **nested parentheses** — that is, parentheses within parentheses.

$$7(5(\$2 + \$4))$$

Because it gets hard to read nested parentheses, brackets [ ] are often used instead of parentheses for the outer grouping when there's an inner pair of parentheses. Below is another way of writing 7(5(\$2 + \$4)).

$$7[5(\$2 + \$4)]$$

Whether written with brackets or two sets of parentheses, we can simplify nested groupings the same way. We need to **start with the innermost grouping and work our way out.**

Example: 7[5(\$2 + \$4)]

Step 1: Solve the innermost grouping.

7[5(**\$2 + \$4**)]

7[5(**\$6**)]

We've now found the cost per person to bowl: \$6.

Step 2: Solve the next grouping.

7[**\$30**]

We've now found the cost for the entire family to bowl each time they go: \$30.

Step 3: Complete the final multiplication.

\$210

We've now found the cost for the entire family to bowl 7 times a year: \$210.

The family should budget \$210.

Think carefully about the example we just solved. Calculating how much it would cost for the entire family to bowl is the type of problem you've been solving since elementary school. Only in elementary school, you would have solved this type of problem as three separate problems, like this:

cost per person to bowl = \$2 + \$4 = \$6

cost for the entire family of 5 to bowl each time they go = 5 • \$6 = \$30

cost for the entire family to bowl 7 times = 7 • \$30 = \$210

Using nested parentheses, we expressed all of these steps as one problem:

cost for the entire family to bowl 7 times = 7[5(\$2 + \$4)]

Keeping Perspective

Parentheses are part of the language side of mathematics. We use them to group operations that need to be performed before others.

As we mentioned back in Lesson 1.2, other methods have also been used to indicate groupings instead of parentheses. Two different historical ways of writing (4 + 5)3 are shown here.[1]

$\overline{4 + 5} \cdot 3$ $\qquad$ $4 + 5, 3$

While the quantities and consistencies God created and sustains remain constant, the symbols we use to describe them can and do vary. God gave man the ability to think creatively and adopt different methods for describing His creation.

As you study the language side of mathematics, remember that you are trying to learn a language to help you clearly describe the quantities and consistencies God created and sustains.

9.2 Parentheses, the Distributive Property, and Unknowns

We saw in the last lesson that parentheses help us group numbers on which operations need to be completed first.

We looked at the formula for converting between Fahrenheit and Celsius:

$$C = \frac{5}{9}(F - 32°)$$

In the last lesson, we only used the formula to convert from Fahrenheit to Celsius. Thus, we always knew what F in the formula represented. But what if we want to convert a temperature in Celsius to Fahrenheit? Could we use the formula to find F if we know C?

For example, if we know it's 20 degrees Celsius, can we find the degrees Fahrenheit?

$$20° = \frac{5}{9}(F - 32°)$$

Yes, we can find the value of F the same way we would the value of other unknowns — by isolating it on a side of the equation by itself. But how do we do that? After all, we have to subtract 32° from F . . . but we don't know F's value!

It's time to review the distributive property and look at how it can help us solve problems like this one.

Applying the Distributive Property

Distributive Property Review

The distributive property is a name to describe the fact that if you distribute multiplication among addends and then add those results together, you'll get the same result as if you first did the addition and then multiplied the sum.

Example: Find $2(3 + 4)$

With distribution:	Adding first:
$2(3 + 4)$	$2(3 + 4)$
$2(\mathbf{3}) + 2(\mathbf{4})$	$2(\mathbf{7}) = 14$
$\mathbf{6} + \mathbf{8} = 14$	

No matter whether we distribute the multiplication among addends or add the addends together first, the answer is the same.

Why? Because of the consistent way addition and multiplication work. The distributive property expresses a reality about the consistency of addition and multiplication God holds in place. The distributive property ultimately works because of God's faithfulness in holding all things together consistently.

Okay . . . so how does the distributive property help us find unknowns? Let's take a closer look. We'll do so with a simple percent example, and then return to temperature.

Example: Suppose your purchase totaled $50. You know you bought two items, one of which cost $30 before tax. You don't remember the cost of the other item, but you know you were charged 6% tax on both items. How much was the other item?

Multiplying by 1.06 is the same as finding 100% plus the 6% tax, as 100% + 6% = 106%, or 1.06. See Lesson 9.4 of Book 1 for details.

Let's write this problem out, using x to stand for the item of which we want to find the price.

$$1.06(\$30) + 1.06(x) = \$50 \quad \textit{or} \quad 1.06(\$30 + x) = \$50$$

In the equation on the left, we're multiplying each item by 1.06 to find its total with tax, and *then* adding them together to find the total.

In the equation on the right, we're adding the items we purchase and then multiplying their total to find the total with tax.

Both equations are two different ways of thinking about the problem. Let's take a closer look, though, at how to solve $1.06(\$30 + x) = \$50$, as you need to know how to handle problems written this way. While there's an unknown inside the parentheses, we can use the distributive property to help us solve this.

Distribute the multiplication:

$$1.06(\$30 + x) = \$50$$

$$1.06\mathbf{(\$30)} + 1.06\mathbf{(}x\mathbf{)} = \$50$$

Notice that now we are multiplying each of the items by 1.06 to find its total with tax rather than adding them together first. In other words, we arrived at the other approach to the problem we mentioned up front!

Simplify:

$$\mathbf{\$31.80} + \mathbf{1.06}x = \$50$$

Now we can solve to find x by isolating it on one side of the equation.

Add –$31.80 to both sides:

$$\$31.80 \mathbf{+ -\$31.80} + 1.06x = \$50 \mathbf{+ -\$31.80}$$

$$\mathbf{1.06}x \mathbf{= \$18.20}$$

Divide both sides by 1.06:

$$\frac{1.06x}{\mathbf{1.06}} = \frac{\$18.20}{\mathbf{1.06}}$$

$$\frac{\cancel{1.06}x}{\cancel{1.06}} = \frac{\$18.20}{1.06}$$

$$\boldsymbol{x} \mathbf{= \$17.17}$$

Notice that if we substitute $17.17 into the original equation, we find that the math checks out.

$$1.06(\$30 + x) = \$50$$

$$1.06(\$30 + \mathbf{\$17.17}) = \$50$$

$$1.06(\mathbf{\$47.17}) = \$50$$

$$\mathbf{\$50} = \$50$$

Technically, we get $50.0002 when we use a value of $17.17 for x. The difference is due to rounding ($17.17 is a rounded answer).

Distributing the multiplication enabled us to solve a problem we otherwise couldn't have.

Let's take a look at applying this property when we have *subtraction* inside the parentheses, such as in $4(2 - x) = 10$.

The distributive property only works on *addends* that make up a sum. Remember, though, that we can **view subtraction as an addition of a negative number.** For example, we can view $4(2 - x) = 10$ as $4(2 + -x) = 10$. We can now use the distributive property to find the unknown!

Distribute the multiplication and simplify:

$$4(2 + -x) = 10$$

$$4(\mathbf{2}) + 4(\mathbf{-x}) = 10$$

$$\mathbf{8 + -4x} = 10$$

Notice that we simplified 4(–x) to –4x. Remember, we can think of –x as (–1)x. So we can think of 4(–x) as 4(–1)(x), which we can simplify to –4x.

Add –8 to both sides:

$$8 + -4x + \mathbf{-8} = 10 + \mathbf{-8}$$

$$-4x = 2$$

Divide both sides by –4:

$$\frac{-4x}{\mathbf{-4}} = \frac{2}{\mathbf{-4}}$$

$$\frac{-4x}{-4} = \frac{2}{-4}$$

$x = -\frac{2}{4}$, which simplifies to $-\frac{1}{2}$

We often view subtraction as an addition of a negative number without writing out the + sign. The key is to view the negative sign as part of the number that follows it.

Example: Suppose that you know you left the store having spent a total of $20. You purchased a $33 item and returned an item, but you can't remember the price of the returned item. You also know you were charged 6% tax on the purchase, and credited 6% on the return.

Using x to stand for the returned item, we could represent this mathematically like this:

$$1.06(\$33 - x) = \$20$$

Think of the parentheses as $(\$33 + -x)$ and distribute the multiplication:

$$1.06(\$33 + \mathbf{-x}) = \$20$$

$$1.06(\$33 + -x) = \$20$$

$$1.06\mathbf{(\$33)} + 1.06\mathbf{(-x)} = \$20$$

Simplify:

$$\mathbf{\$34.98 + -1.06x} = \$20$$

Add –$34.98 to both sides:

$$\$34.98 + \mathbf{-\$34.98} + -1.06x = \$20 + \mathbf{-\$34.98}$$

$$\mathbf{-1.06x = -\$14.98}$$

Again, we could also have written this problem as $1.06(\$33) + 1.06(-x) = \$20$. After all, we could find the cost with tax of each item and then add them together instead of adding them together and then finding the tax. Notice that when we distribute the multiplication, we end up with this equation.

Divide both sides by –1.06:

$$\frac{-1.06x}{\mathbf{-1.06}} = \frac{-\$14.98}{\mathbf{-1.06}}$$

$$\frac{\cancel{-1.06}x}{\cancel{-1.06}} = \frac{-\$14.98}{-1.06}$$

$$\boldsymbol{x = \$14.13}$$

Again, if we substituted \$14.13 into the original equation for x, we can verify that we've successfully found the value for x.

$$1.06(\$33 - x) = \$20$$

$$1.06(\$33 - \mathbf{\$14.13}) = \$20$$

$$1.06\mathbf{(\$18.87)} = \$20$$

$$\mathbf{\$20} = \$20$$

\$20 dollars does equal \$20. We found the correct value for x.

Technically, we get \$20.0022 when we use a value of \$14.13 for x. The difference is due to the rounding we did when solving the problem.

Distributive Property and the Temperature Formula

Now that we have reviewed the distributive property and used it to find the unknown in a couple of everyday examples, it's time to return to the temperature problem we began the chapter with: converting 20 degrees Celsius to Fahrenheit. Now we can easily solve this using the distributive property.

Example: Convert 20 °*C* to °*F*.

Formula for converting between Celcius and Fahrenheit:

$$C = \frac{5}{9}(F - 32°)$$

Substitute known values:

$$\mathbf{20°} = \frac{5}{9}(F - 32°)$$

Distribute the multiplication, being careful to view –32° as a negative number:

$$20° = \frac{5}{9}(F + -32°)$$

$$20° = \frac{5}{9}(F + -32°)$$

$$20° = \frac{5}{9}\mathbf{(}F\mathbf{)} + \frac{5}{9}\mathbf{(-32°)}$$

Simplify:

$$20° = \frac{5}{9}\boldsymbol{F} + \mathbf{-\frac{160°}{9}}$$

Add $\frac{160}{9}°$ to both sides of the equation:

$$20° + \mathbf{\frac{160°}{9}} = \frac{5}{9}F + -\frac{160°}{9} + \mathbf{\frac{160°}{9}}$$

$$\mathbf{\frac{340°}{9}} = \frac{5}{9}F$$

Divide both sides by $\frac{5}{9}$, which we do by multiplying by $\frac{9}{5}$:

$$\frac{340°}{9} \bullet \frac{\mathbf{9}}{\mathbf{5}} = \frac{5}{9}F \bullet \frac{\mathbf{9}}{\mathbf{5}}$$

$$\overset{68}{\cancel{\frac{340°}{9}}} \bullet \frac{\cancel{9}}{\cancel{5}} = \frac{\cancel{5}}{\cancel{9}}F \bullet \frac{\cancel{9}}{\cancel{5}}$$

$$\mathbf{68°} = \boldsymbol{F}$$

20° Celsius equals 68° Fahrenheit.

We could check the answer by substituting 68° for *F* in the original equation:

$$20° = \frac{5}{9}(\mathbf{68°} - 32°)$$

$$20° = \frac{5}{9}(\mathbf{36°})$$

$$20° = \mathbf{20°}$$

Now I know this example may look hard, but each step is not hard. Complicated-looking problems are simple when you take them step by step.

Keeping Perspective

Notice that once again, we're just applying the *same principles* we learned in basic math to help us work with unknown as well as known numbers. We didn't cover anything *new* today — rather, we explored how to apply what we had already been doing with known numbers to unknowns.

As you use the distributive property with unknowns, remember that the distributive property merely describes a consistency that holds true with multiplication. And why does it hold true? Because of the consistent way God governs this universe!

9.3 Positive Signs, Parentheses, and the Distributive Property

As we've seen, parentheses are a useful grouping tool. In the last lesson, we used the distributive property to help us remove parentheses when an unknown was inside of them. In this lesson, we're going to continue looking at parentheses, the distributive property, and unknowns, focusing on problems in which a parenthesis is directly preceded by a positive sign.

For example, let's suppose that you want to find the total you spent over two days. Your total will equal the sum of the total you spent each day.

total spent day 1 + total spent day 2 = total for both days

Let's say your total the first day was \$5, and the second day you spent \$4 and \$36. You'd have this equation:

$$\$5 + (\$4 + \$36) = \textit{total for both days}$$

Notice that we put the \$4 and \$36 inside of parentheses, as the relationship said to add the *total* for each day together. Now, while in this case we could simply complete all the addition and solve the problem, what would we do if the parentheses contained an unknown — if we didn't remember the amount of the \$36 purchase, for instance, but knew the total purchases for both days was \$45?

$$\$5 + (\$4 + x) = \$45$$

Notice that we now have an unknown *inside* the parentheses. How can we solve this? There doesn't appear to be anything to distribute.

Well, we know from the identity property of multiplication that multiplying by 1 doesn't change the value of a quantity. **So we could multiply the parentheses by 1 and distribute the 1 without changing the value.**

Multiply the parentheses by 1:

$$\$5 + 1(\$4 + x) = \$45$$

We can now distribute the 1 the same way we distributed other quantities in the last lesson!

Distribute the multiplication by 1 and simplify:

$$\$5 + 1(\$4 + x) = \$45$$
$$\$5 + 1\mathbf{(\$4)} + 1\mathbf{(}\boldsymbol{x}\mathbf{)} = \$45$$
$$\$5 + \mathbf{\$4} + \boldsymbol{x} = \$45$$

We were able to remove the parentheses by multiplying the parentheses by 1. However, we didn't need to write out the distribution. Notice that we would have arrive at $\$5 + \$4 + x = \$45$ by simply removing the parentheses.

Removing the parentheses from

$$\$5 + (\$4 + x) = \$45$$

gives us

$$\$5 + \$4 + x = \$45$$

When a set of parentheses is preceded by a + sign, multiply the values inside the parentheses by 1 and distribute the 1. You can do this by simply removing the parentheses. After all, multiplying by positive 1 doesn't change the value, so distributing a positive 1 isn't going to change the value.

Once the parentheses have been removed, we can easily solve to find x.

Simplify:

$$\$5 + \$4 + x = \$45$$
$$\mathbf{\$9} + x = \$45$$

Add –$9 to both sides of the equation:

$$\$9 + x + \mathbf{-\$9} = \$45 + \mathbf{-\$9}$$
$$\boldsymbol{x} = \mathbf{\$36}$$

Keeping Perspective

While today's lesson may have felt a little obvious, sometimes it's worth stating the obvious. It's important to understand that parentheses are a grouping symbol that tells us to solve whatever is inside the parentheses first. We can only remove parentheses after we've performed the operations inside them *or* if we know that removing the parentheses will preserve the same meaning (such as if we're able to distribute the multiplication).

9.4 Parentheses in Action

We've been spending a lot of time lately looking at the mechanics of simplifying expressions involving unknowns. It's time to take a break from the mechanics to remind ourselves why we're learning these simplification techniques: so we can solve real-life problems.

In fact, we're going to take the entire lesson to walk through one problem. While this problem is more in-depth than any of the ones we've looked at so far, it's a good illustration of the type of real-life problems you'll be able to solve once you get familiar with the mechanics we've been exploring.

Example: At a local election there were only 2 candidates, and 1,280 votes were cast. The winning candidate won by only 40 votes. How many votes were cast for each candidate?

It might initially look as if we don't have enough information to answer the question. After all, we don't know the total votes cast for *either* candidate. We have *two unknowns* right now — the votes cast for the loser and the votes cast for the winner.

All we know is that the sum of the votes equals 1,280, and that the votes for the winner equal 40 more than the votes for the loser.

1,280 = *votes for the loser* + *votes for the winner*

votes for the winner = *votes for the loser* + 40

If we use x to stand for the votes for the loser, we could describe the votes for the winner as $x + 40$. This would give us the following:

This problem is adapted from a math book[2] from the early 1900s.

votes for the loser = x

votes for the winner = $x + 40$

We know the total votes cast here were 1,280. Since there were only 2 candidates, the votes for the loser plus the votes for the winner must equal 1,280.

1,280 = *votes for the loser* + *votes for the winner*

1,280 = $\boldsymbol{x + (x + 40)}$

Do you see what we just did? **We found a way to express the problem using only one unknown.**

Notice we put parentheses around (x + 40). Why? Because we know that *the sum of* x + 40 represents the votes for the winner. So when we substitute x + 40 for the votes for the winner, we used parentheses to show that we were replacing the votes for the winner with the *sum of* x + 40.

Let's solve 1,280 = x + (x + 40) to find x.

Remove the parentheses:

1,280 = $x + (x + 40)$

1,280 = $x + \boldsymbol{x + 40}$

Note: We removed the parentheses because we knew that we could multiply the parentheses by 1, distribute that 1, and the values would be the same, as multiplying by 1 doesn't affect the values of anything inside the parentheses. We didn't bother to write out the steps, since multiplying by 1 doesn't affect the values.

Combine like terms:

1,280 = $\boldsymbol{2x} + 40$

Subtract 40 from both sides:

1,280 $\boldsymbol{-\ 40}$ = $2x + 40 \boldsymbol{-\ 40}$

$1{,}240 = 2x$

Divide both sides by 2 to find x:

$$\frac{1{,}240}{\boldsymbol{2}} = \frac{2x}{\boldsymbol{2}}$$

$$\overset{620}{\cancel{\frac{1{,}240}{2}}} = \cancel{\frac{2x}{2}}$$

$\boldsymbol{620 = x}$

Now remember, x represented the votes for the loser. So now we can find the vote for the winner by simply substituting the value we found!

votes for the winner = $x + 40$

Now that we know x = 620, this would mean:

votes for the winner = **620** + 40

votes for the winner = **660**

We'll check our math by substituting values back into the original relationships:

votes for winner = votes for the loser + 40
660 = **620** + 40
660 = **660**

total votes = votes for the loser + votes for the winner
1,280 = **620 + 660**
1,280 = **1,280**

We just solved a problem that had *two* missing pieces of information. By looking at how those pieces *related to each other*, we were able to find one missing piece and then use it to find the other.

There's More Than One Way to Peel an Onion (or Solve a Problem)

In the example we just explored we could have used *x* to represent votes for the winner rather than the votes for the loser.

If so, we would show the relationship this way:

votes for the winner = x
votes for the loser = x – 40
Total = votes for loser + votes for winner
1,280 = (*x* – 40) + *x*

If we complete this problem, we'll come out with the same values for the winner and loser we did before. Only this time, *x* will represent the winner.

1,280 = **2*x*** – 40

Add 40 to both sides:

1,280 **+ 40** = 2*x* – 40 **+ 40**
1,320 = 2*x*

Divide both sides by 2 to find *x*:

$$\frac{1{,}320}{\mathbf{2}} = \frac{2x}{\mathbf{2}}$$

$$\frac{1{,}320}{2} = \frac{2x}{2} \quad (660)$$

660 = *x*

Since we used *x* to represent the votes for the *winner*, we need to substitute to find the votes for the *loser*.

votes for loser = x – 40
votes for loser = **660** – 40
votes for loser = **620**

We arrived at the same answers as before: 660 votes for the winner and 620 votes for the loser. It doesn't matter which unknown you solve for, so long as you keep the relationships accurate and are consistent in your use of the unknown.

Keeping Perspective

While you won't be required to solve problems with two unknowns without lots of hints right now, I hope the example we looked at gave you a glimpse of how the skills you're learning can be used. Remember, math isn't a bunch of meaningless rules or numbers. It's a way of describing real-life consistencies God created and sustains. As such, it applies outside of a textbook! You can use the skills you're learning in math to help you in whatever occupation or sphere of influence to which God calls you.

9.5 Parentheses, the Distributive Property, Unknowns, and Negative Numbers

Because grouping numbers is so very helpful, especially as equations get more and more in-depth, we need to take one more lesson to look at applying the distributive property to help us deal with parentheses containing unknowns. Specifically, we're going to take a look at situations involving negative numbers. Once again, we'll be applying what we already know to help us in new situations.

Parentheses Preceded by a Negative Number

Distributive Property and Negative Numbers

In order to properly distribute the multiplication when the multiplier is preceded by a negative sign, **view the multiplier as a negative number.**

Example: Find 5 – 2(3+4)

With distribution:	Adding first:
$5 + (-2)(3 + 4)$	$5 - 2(3 + 4)$
$5 + \ -2\mathbf{(3)} + -2\mathbf{(4)}$	$5 - 2\mathbf{(7)}$
$5 + \ \mathbf{-6} \ + \ \mathbf{-8} \ = \mathbf{-9}$	$5 - \ \mathbf{14} \ \ = \mathbf{-9}$

Viewing a multiplier preceded by a negative sign as a negative number preserves the intended meaning. For example, the expression 5 – 2(3 + 4) means to subtract 2 times the total inside the parentheses from 5. If we don't distribute the negative sign,[3] we'll end up only subtracting *part* of the total, which gives us an *incorrect* answer.

Notice how in the example below, instead of subtracting 6 and 8, we end up subtracting 6 and *adding* 8, thus getting an incorrect answer.

Incorrect:

$5 - 2(3 + 4)$

$5 - 2\mathbf{(3)} + 2\mathbf{(4)}$

$5 - \mathbf{6} + \mathbf{8} = \mathbf{6}$

I can't say it enough times: **train yourself to look at subtraction as the addition of negative numbers.**

Let's apply the distributive property to unknowns. Say someone earned $200, and then bought 2 DVDs at $14 each plus 2 shirts, the price of which he can't remember. He now has $100 left of the money he earned. How much did he spend on each shirt?

Let's write this out, using x to stand for the price of each shirt:

$$\$200 - 2(\$14) - 2x = \$100 \quad \textit{or} \quad \$200 - 2(\$14 + x) = \$100$$

In the equation on the left, we're subtracting the costs of the 2 DVDs and the 2 shirts separately. In the equation on the right, we're adding the cost of each DVD and shirt together, multiplying it by 2, and then subtracting that product from $200. Both are different ways of approaching the same problem.

Let's take a closer look at how to solve $\$200 - 2(\$14 + x) = \$100$, as you need to know how to handle problems written this way. Notice there's an unknown inside the parentheses, so we *can't* simply complete the addition and then multiply. We can, however, distribute the multiplication.

Example: $\$200 - 2(\$14 + x) = \$100$

Distribute −2:

$$\$200 - 2(\$14 + x) = \$100$$

$$\$200 + \mathbf{-2(\$14)} + \mathbf{-2(x)} = \$100$$

Notice that we are now subtracting the costs of the DVDs and the shirts separately, rather than adding them together first. In other words, we arrived at the other approach to the problem we mentioned up front.

Simplify:

$$\$200 + \mathbf{-\$28} + \mathbf{-2x} = \$100$$

$$\mathbf{\$172} + -2x = \$100$$

Add −$172 to both sides of the equation:

$$\$172 + \mathbf{-\$172} + -2x = \$100 + \mathbf{-\$172}$$

$$\mathbf{-2x = -\$72}$$

Divide both sides of the equation by −2:

$$\frac{-2x}{\mathbf{-2}} = \frac{-\$72}{\mathbf{-2}}$$

$$\frac{\cancel{-2}x}{\cancel{-2}} = \frac{\cancel{-}\$72^{\$36}}{\cancel{-2}}$$

$$\boldsymbol{x = \$36}$$

Each shirt cost $36.

We can check the answer by substituting $36 for x in the original equation:

$$\$200 - 2(\$14 + \mathbf{\$36}) = \$100$$

$$\$200 - 2\mathbf{(\$50)} = \$100$$

$$\$200 - \mathbf{\$100} = \$100$$

$$\mathbf{\$100} = \$100$$

Parentheses Preceded by a Negative Sign

Sometimes, all we may have in front of a set of parentheses is a negative sign.

For example, suppose we knew that a game of bowling cost $2 in shoe rental, but we couldn't remember the cost per game. We knew we'd started with $8 in our wallet, and ended with $3 after renting shoes and playing one game of bowling. We'd have this problem:

$$\$8 - (\$2 + x) = \$3$$

Notice that we've used parentheses to group the total costs of the shoe rental and the cost per game (x). We want to subtract the total cost from $8, as $8 minus the total we spent will equal the $3 left in our wallet.

But how can we solve this since we don't know the cost per game (x)?

Let's take a look.

Example: $\$8 - (\$2 + x) = \$3$

As we've seen, multiplying by 1 doesn't change the value of a number (the identity property of multiplication). So let's multiply the values inside the parentheses by 1.

$$\$8 - 1(\$2 + x) = \$3$$

Now we have an equation we know how to solve! We can distribute the –1, just as we did the –2 in the previous section.

To clarify, we'll add a plus sign before the –1, as we have to be sure to view the –1 as a negative number in order to properly distribute the multiplication.

$$8 + \mathbf{-1}(2 + x) = 3$$

Distribute –1 and simplify:

$$8 + (-1)(2 + x) = 3$$

$$\$8 + \mathbf{-1(\$2)} + \mathbf{-1(}\boldsymbol{x}\mathbf{)} = \$3$$

$$\$8 + \mathbf{-\$2} + \mathbf{-}\boldsymbol{x} = \$3$$

$$\mathbf{\$6} + -x = \$3$$

Add –6 to both sides:

$\$6 + \mathbf{-\$6} + -x = \$3 + \mathbf{-\$6}$

$\mathbf{-x} = \mathbf{-\$3}$

Multiply both sides by –1:

$(-x)\mathbf{(-1)} = (-\$3)\mathbf{(-1)}$

$\boldsymbol{x} = \mathbf{\$3}$

Check the answer by substituting \$3 for x in the original equation:

$\$8 - (\$2 + \mathbf{\$3}) = \$3$

$\$8 - (\mathbf{\$5}) = \$3$

$\mathbf{\$3} = \$3$

This is true. We found the correct value for x.

Distributing multiplication between addends — including distributing a negative sign — preserves the intended meaning. Both $8 – ($2 + $3) and $8 + –$2 + –$3 mean to subtract a total of $5 from $8. When we have an unknown inside the parentheses, distributing the multiplication can help us solve what we otherwise wouldn't be able to.

When a set of parentheses is preceded by a – sign, distribute the – sign by multiplying the values inside the parentheses by 1 and distributing –1.

Example: Simplify $-(2 + x)$

$\mathbf{-1}(2 + x)$

$(-1)\mathbf{(2)} + (-1)\mathbf{(x)}$

$\mathbf{-2} + \mathbf{-1x}$, or $-2 + -x$

Understanding that you can multiply any quantity by 1 without changing its value (the identity property of multiplication) is crucial. In fact, it can be helpful to think of there being an understood 1 next to every unknown or parentheses.

$a = \mathbf{1}a$ $\qquad$ $-a = \mathbf{-1}a$

$x = \mathbf{1}x$ $\qquad$ $-x = \mathbf{-1}x$

$(x + 5) = \mathbf{1}(\mathbf{1}x + 5)$ $\qquad$ $-(2 + x) = \mathbf{-1}(2 + \mathbf{1}x)$

However you think of it, know that **you can multiply by 1 without changing the value.**

Keeping Perspective

Each new rule in math builds on previous principles or rules, ultimately going back to the underlying consistency God created and sustains. The rules and conventions, like a language system, help us represent and work with that consistency.

9.6 Chapter Synopsis and the Beauty of Mathematics

We've been looking a lot at unknowns in this course and at how to find them. In this chapter, we focused on applying the **distributive property** to find an unknown inside **parentheses** (a grouping).

Once again, we've been applying what we *already know* about known numbers to help us work with unknowns. We can apply the same principles to unknowns that we do to known quantities because of the unerringly consistent way God holds all things together.

While we haven't used the word, as we've been exploring unknowns and how to work with them, we've really been learning the core principles of the branch of math called algebra. While the word "algebra" sounds intimidating, **algebra** is simply "the part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations"[4] or "the science which teaches how to determine unknown quantities by means of those that are known."[5]

Don't let algebra intimidate you. It's simply the branch of math where we learn to apply the *same principles* we've been using with known quantities to unknowns, thereby being equipped to explore the quantities and consistencies God created and sustains around us to an even deeper level.

The Beauty of Mathematics

Have you ever thought about the beauty God placed around us? He could have made everything all one color — but instead, He wove together a vast variety of hues. He put interest everywhere we look.

Since math reflects the consistencies around us, it shouldn't surprise us to find beauty in math itself. Many pure mathematicians devote their time to studying math for math's own sake. They don't necessarily apply their mathematical knowledge to real life — they simply want to explore the beauty and wonder within numbers themselves. They would say there's something beautiful about certain equations — and in a way, there is!

As creatures made in God's image, we can recognize the beauty of math itself. It's good to recognize the beauty in math at every level, much as we would recognize the beauty of a painting. We should always remember, though, that God is the master artist — if men are able to paint a beautiful picture, it is only because God made us in His image (and our picture is but a copy of the picture God has painted). Math's beauty ultimately points us to God's creativity and care.

[CHAPTER 17]

Square Roots and Unknowns

17.1 Square Roots

Now that we've explored exponents, we're going to spend this chapter looking at a related concept: roots. We'll find that they, like exponents, help us describe and work with the real-life relationships God has placed around us.

Let's start today by making sure we understand what's meant by a square root. In the next lesson, we will begin to apply square roots to unknowns.

Understanding Square Roots

Exponents allow us to represent repeated multiplication. When we *square* a number, we mean to multiply it by itself. A *square root* reverses a squared number. It asks for a number that, times itself, equals the given number.

4 *squared* (written 4^2) = 4 • 4, or 16.

the *square root* of 16 (written $\sqrt{16}$) = 4, since 4 • 4 = 16

Just as 4^2 (4 *squared*) represents the repeated multiplication of 4 • 4, $\sqrt{16}$ (the *square root* of 16) represents the division necessary to reverse that repeated multiplication. A **square root** (often shown with a $\sqrt{\ }$ symbol) is a way of concisely asking for the number that, times itself, will equal the number given.

Two Square Roots

Since a negative number times itself yields a *positive* number, the square root of a positive number could be either positive *or* negative.

$$\sqrt{16} = 4 \text{ or } -4$$

$$4 \bullet 4 = 16 \quad \textit{and} \quad -4 \bullet -4 = 16$$

$$\sqrt{4} = 2 \text{ or } -2$$

$$2 \bullet 2 = 4 \quad \textit{and} \quad -2 \bullet -2 = 4$$

$$\sqrt{64} = 8 \text{ or } -8$$

$$8 \bullet 8 = 64 \quad \textit{and} \quad -8 \bullet -8 = 64$$

In Book 1, you learned to express that a square root could be either positive or negative by writing a ± in front of your answer. We emphasized this point because it's important to realize that there is both a positive and a negative square root for every positive real number.

In algebra and much of upper math, though, it's common to assume that a positive root is meant when you see a root sign ($\sqrt{\ }$) unless told otherwise. This avoids a lot of ambiguity and confusion, especially as we begin exploring more complex relationships.

From now on, unless you're asked to give both the negative and the positive square root, **you can assume the positive square root is meant**. In real life, don't assume a positive square root when a negative answer could make sense.

Finding a Square Root

Think about what we're trying to find when we are finding a square root. We want to find a number that, times itself, equals the given number. In other words, we need to find the *factor* that, times itself, equals the number of which we're finding the square root. Any thoughts as to a technique that can help us find the factors of a number?

Factoring! If we find the prime factors of a number, we can more easily spot the square root.

Example: Find $\sqrt{4}$

Let's start by factoring 4.

$$\begin{array}{c} 4 \\ \wedge \\ 2 \bullet 2 \end{array}$$

Factoring makes it easy to see that the square root of 4 — the number that, times itself, equals 4 — is 2.

Example: Find $\sqrt{36}$

Let's start by factoring 36.

$$\begin{array}{r} 36 \\ \wedge \\ 3 \bullet 12 \\ \quad \wedge \\ \quad 3 \bullet 4 \\ \quad\quad \wedge \\ \quad\quad 2 \bullet 2 \end{array}$$

36 expressed as the product of its prime factors: $3 \bullet 3 \bullet 2 \bullet 2$

The square root isn't instantly obvious this time. But notice that it takes two 3s and two 2s multiplied together to make 36. We're looking for a number that, times itself, equals 36. We can clump prime factors together to find a factor that, times itself, will equal 36. Our answer would be $2 \bullet 3$, or 6.

$(2 \bullet 3) \bullet (2 \bullet 3) = 36$

$6 \bullet 6 = 36$

$\sqrt{36} = 6$

It's not always necessary to find all the prime factors. Sometimes you'll know the square root just by thinking through your multiplication facts. In fact, with 36, you knew $6 \bullet 6 = 36$. However, knowing how to find the square root by factoring is both helpful in understanding square roots and useful for finding square roots you don't know.

Keeping Perspective

While square roots should be review for you, we'll begin exploring in the next lesson how square roots aid us in finding unknowns. Like exponents, we'll find square roots quite useful in working with real-life mathematical relationships.

17.2 Square Roots and Unknowns (and Squaring Both Sides of an Equation)

Now that we've reviewed square roots, it's time to look at how to use them with unknowns. Once again, we'll find that the *same principle* that applies to known quantities applies to unknowns as well.

Square Roots of Squares

Consider this expression:

$$\sqrt{4^2}$$

Any idea what it means? Well, it's asking for the square root of 4^2. Since 4^2 equals 16, $\sqrt{4^2}$ is another way of writing $\sqrt{16}$.

$$\sqrt{4^2} = \sqrt{16}$$

The answer to both of these expressions is 4, as 4 • 4 equals 16.

Notice that we could have easily seen the answer just by looking at $\sqrt{4^2}$. If the square root of a number is the number that, times itself, equals the number, then the square root of 4^2 is going to be 4. The answer is obvious if we remember the meaning of square roots and exponents: we know that the square root of a squared number will be the base that's being squared.

$$\sqrt{8^2} = 8$$

$$\sqrt{10^2} = 10$$

$$\sqrt{115^2} = 115$$

This holds true for unknowns too. In the following expressions, it doesn't matter that we don't know the value for the unknown. We know the square root of the unknown squared will be the unknown itself.

$$\sqrt{v^2} = v$$

$$\sqrt{x^2} = x$$

$$\sqrt{a^2} = a$$

Starting in the next lesson, we'll use this concept to help us solve many problems. Taking the square root of numbers that are squared proves quite useful.

Squaring Both Sides of an Equation

Now let's say that we know that $\sqrt{x} = 4$. How do we find x?

Well, think about what the square root sign means. It indicates the number that, times itself, equals a given number. If we know the square root of x is 4, we know that x must be 4 • 4, or 16. After all, $\sqrt{16} = 4$.

Now, let's think about how we could have found that mathematically. Just as we can multiply both sides of an equation by the same quantity without changing the equality, we can square both sides of an equation without changing the equality.

Example: $\sqrt{x} = 4$. Find x.

Square *both sides* of the equation:

$(\sqrt{x})^2 = (4)^2$

What does $(\sqrt{x})^2$ mean? Well, we know $\sqrt{x} \bullet \sqrt{x}$ has to equal x, as a square root means a number that, times itself, equals the quantity under the $\sqrt{\ }$ sign.

$x = 16$

Keeping Perspective

The more you learn in math, the more you'll keep building on the same basic principles. In this lesson, we built on the concept of square roots, exponents, and equality to help us find the value of an unknown under a square root symbol. All of math rests on the fundamental principles you've been learning since elementary school, which in turn rest on God's faithfulness in holding all things together.

17.3 Taking the Square Root of Both Sides

You've learned quite a bit so far about finding unknowns. For instance, you've learned that adding, subtracting, multiplying, or dividing the *same amount* from *both sides* of an equation doesn't alter the equality. The equation stays in balance.

It's time now to dig a little deeper and build on the principles you know to work with relationships involving square roots.

Taking the Square Root of Both Sides of an Equation

Just as we can divide both sides of an equation by the same quantity without changing the equality, we can also take the square root of both sides of an equation without changing the equality.

Example: $64 = 64$

$\sqrt{64} = \sqrt{64}$

8 = 8

Taking the square root of *both sides* of an equation proves most useful in solving for unknowns.

Example: $64 = x^2$

How do we find x? Well, if we take the square root of both sides, we'll find the value of x. After all, we saw in the last lesson that the square root of a squared number is the number itself, so the square root of x^2 is x.

$$\sqrt{64} = \sqrt{x^2}$$

$$\mathbf{8 = x}$$

Let's apply this concept to a practical example. We know that the area of a square equals the length of one of its sides squared.

$$Area = side^2 \quad \text{or} \quad A = s^2$$

So let's say we want to figure out how long to make the sides of a square if we want it to take up an area of 25 ft^2.

Example: Find the sides of a square with an area of 25 ft^2.

Substitute known values:

$$A = s^2$$

$$25 \text{ ft}^2 = s^2$$

Now, how do we find the value of s? By taking the square root of both sides of the equation! That will tell us the number that, times itself, equals 25 ft^2.

$$\sqrt{25 \text{ ft}^2} = \sqrt{s^2}$$

$$\mathbf{5 \text{ ft} = s}$$

Notice that when we found $\sqrt{25 \text{ ft}^2}$, we ended up with 5 ft. Don't let the units confuse you. Just as the square root of s^2 is s, so the square root of ft^2 is ft. After all, if we multiply 5 ft by 5 ft, we do indeed end up with 25 ft^2.

Back in Book 1, you learned this formula for finding the length of each side of a square when you knew the area:

$$\sqrt{Area} = side \quad \text{or} \quad \sqrt{A} = s$$

We mentioned then that you'd learn in Book 2 how we came up with this formula. Well, that time has come.

We came up with this formula by taking the square root of both sides of the formula for finding the area of a square.

Formula for finding the area of a square:

$$A = s^2$$

Take the square root of *both sides:*

$$\sqrt{A} = \sqrt{s^2}$$

Simplify (we know the square root of s^2 will be s):

$$\sqrt{A} = \boldsymbol{s}$$

It doesn't matter if we're dealing with known numbers or unknowns — we can take the square root of *both sides* of an equation.

Keeping Perspective

In this lesson, we saw that we can **take the square root of both sides of an equation**. As you begin working with more involved mathematical relationships, you'll use the principle of performing the *same operation* of the *same amount* to both sides of an equation over and over again to find unknowns. As you do, remember that the whole reason we can work with mathematical relationships and unknowns to begin with is because of the consistent way God holds all things in place.

17.4 Approximating Square Roots and Finding Square Roots on a Calculator

As we apply square roots, we'll frequently encounter numbers that don't have an integer square root.

For example, what is $\sqrt{65.5}$? We know that it's more than 8, as 8 • 8 equals 64. But it's less than 9, as 9 • 9 equals 81.

So the $\sqrt{65.5}$ is between 8 and 9.

If we needed a more exact answer, we could estimate some different values until we narrowed down the square root more precisely. For example, we might see if $\sqrt{65.5}$ is 8.10.

$$8.10 \bullet 8.10 = 65.61$$

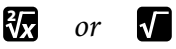
Hmm . . . we're very close, but $\sqrt{65.5}$ is less than 8.10, as we want the number that, when squared, equals 65.5, not 65.61. So let's try a number slightly less than 8.10, such as 8.09.

$$8.09 \bullet 8.09 = 65.45$$

Well, 8.09 was too low. $\sqrt{65.5}$ is somewhere between 8.09 and 8.10.

We could keep trying numbers to pin $\sqrt{65.5}$ down, or we could use a calculator to do the calculations for us.

To find a square root on a calculator, look for the square root button, which should look something like this:

 *or*

Square Root Button

(The x in the button on the left is just a placeholder to represent whatever quantity is under the square root sign.)

Example: Find $\sqrt{65.5}$

Enter 65.5 into the calculator.

Press $\sqrt[3]{x}$ or $\sqrt{\ }$.

$\sqrt{65.5} = 8.0932070281\ldots$

Note that the calculator will automatically give the positive square root.

On some calculators, you'll have to press the square root button *first*, and then enter the number of which you want to find the square root. Take some time to be sure you know how to find the square root on your calculator.

We focused this lesson on learning the skill of approximating square roots and finding them on a calculator. The purpose of familiarizing ourselves with square roots is to be equipped to apply them. For instance, say we wanted the area of a square garden to be 65.5 ft^2. What length should we make each side? We could answer the question by finding the square root of 65.5 ft^2 . . . which we learned how to do both through approximating or using a calculator.

Keeping Perspective

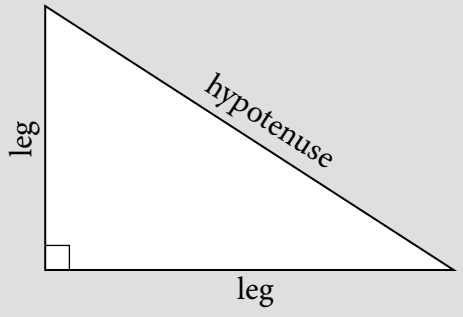
Worksheet 17.4 will help you practice both approximating square roots and finding them on a calculator. While the calculator will obviously be easier, approximating square roots will help you become familiar with square roots. The goal is to be equipped to use square roots as a tool, no matter what type of number you encounter.

17.5 Pythagorean Theorem

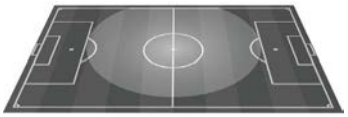
It's time now to apply square roots to help us explore a handy relationship related to right triangles. To understand this relationship, though, we have to first learn about right triangles.

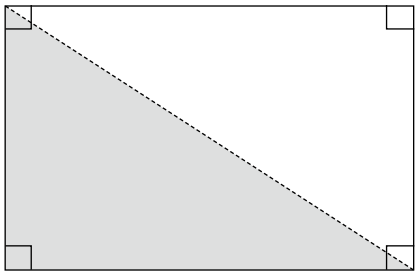
Right Triangles

A **right triangle** is a triangle with a right angle (a 90° angle, which we mark with a ⌝, as shown in the picture). In a right triangle, we call the two sides forming the right angle **legs**, and the side opposite the right angle the **hypotenuse**.



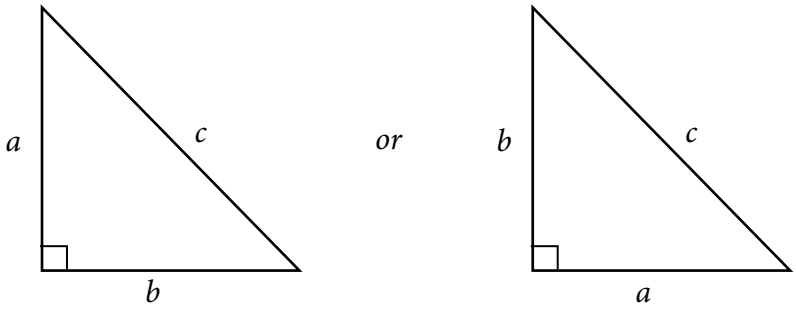
Right triangles prove incredibly useful in describing all sorts of distances — some of which you wouldn't at first suppose they could. For example, we can look at a rectangle (such as a soccer field, bookcase, door, etc.) as two right triangles put together.





Understanding the Relationship

In a right triangle, if we square all the sides, the square of the hypotenuse will equal the sum of the squares of the other two sides. This relationship is known as the **Pythagorean theorem**. It is typically expressed like this:



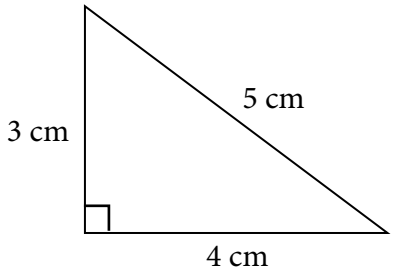
Pythagorean Theorem:

$a^2 + b^2 = c^2$

where c stands for the length of the hypotenuse of a right triangle, and a and b stand for the length of the two legs of the same right triangle

Don't let the letters a, b, and c confuse you. It doesn't matter which leg we view as a and which we view b — the Pythagorean theorem is simply stating that if we square both legs and add them together, it equals the square of the hypotenuse. This relationship has to hold true in order to form a triangle with a right angle.

Let's take a look at a triangle and see the theorem in action. The sides of the right triangle below are 3, 4, and 5 centimeters (you can measure them if you like).



If we were to substitute these measurements into $a^2 + b^2 = c^2$ and simplify, notice that the Pythagorean theorem holds true.

Substitute known values:

$$a^2 + b^2 = c^2$$

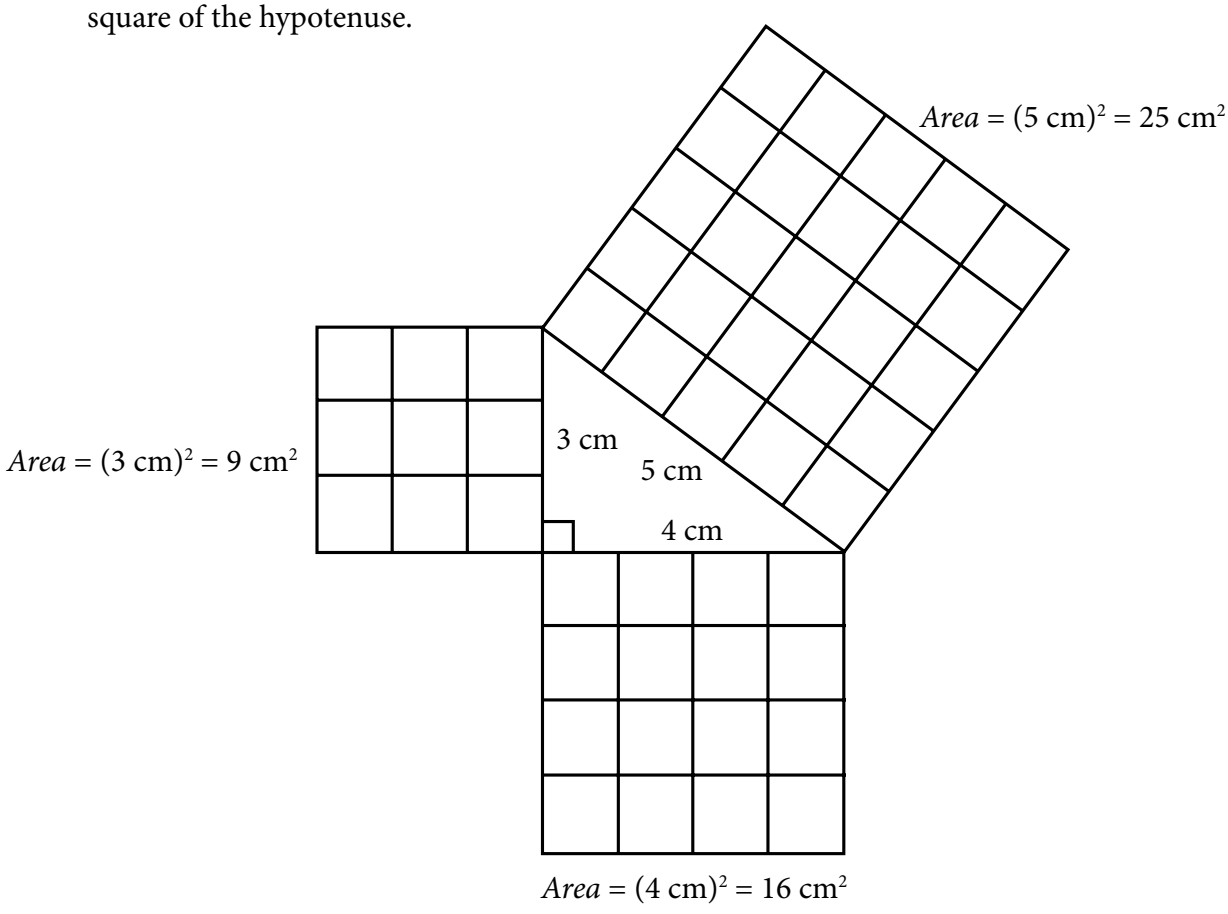
$$\mathbf{(3\ cm)^2 + (4\ cm)^2 = (5\ cm)^2}$$

Simplify:

$$\mathbf{9\ cm^2 + 16\ cm^2 = 25\ cm^2}$$

$$\mathbf{25\ cm^2} = 25\ cm^2$$

The following picture shows this visually. In it, we've built actual squares off each side. The sum of the area of the squares built off the two legs equals the sum of the area of the square built off the hypotenuse. The square of both legs equals the square of the hypotenuse.



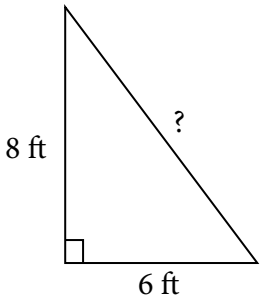
$$\mathbf{9\ cm^2 + 16\ cm^2 = 25\ cm^2}$$

Using the Relationship

You may be wondering why we care that, in a right triangle, the square of the sides equals the square of the hypotenuse. Well, it can help us find the length of a missing side.

Say we know the lengths of the legs of a right triangle, but don't know its hypotenuse. We can use the knowledge of the relationship between the legs and the hypotenuse (i.e., the Pythagorean theorem) to figure out the missing dimension.

Example: Use the Pythagorean theorem to find the missing side of this triangle.



Pythagorean theorem:

$a^2 + b^2 = c^2$

Substitute known lengths — we know the lengths of the two legs (a and b — again, it doesn't matter which leg we view as a and which one we view as b . . . the point is simply to add their squares together):

$\mathbf{(6\ ft)}^2 + \mathbf{(8\ ft)}^2 = c^2$

Simplify:

$\mathbf{36\ ft^2 + 64\ ft^2} = c^2$

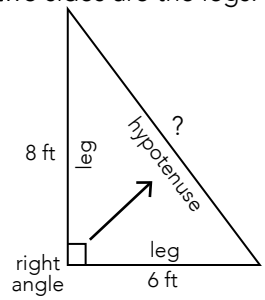
$\mathbf{100\ ft^2} = c^2$

Now we've found c^2, but we still need to find c. We can easily do this by taking the square root of both sides of the equation.

$\sqrt{100\ \text{ft}^2} = \sqrt{c^2}$

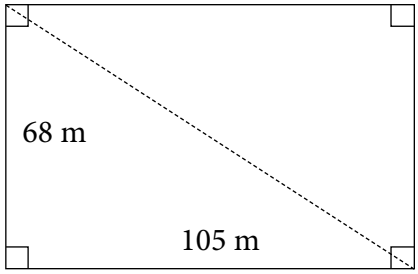
$\mathbf{10\ ft} = \boldsymbol{c}$

How do we know the 8 ft and 6 ft sides are the two legs and the side marked with a question mark is the hypotenuse (represented by c in the formula)? Because the hypotenuse is the side opposite the right angle, while the legs are the sides that form the right angle. Since the side marked with a ? is opposite the right angle, we know it is the hypotenuse, while the other two sides are the legs.



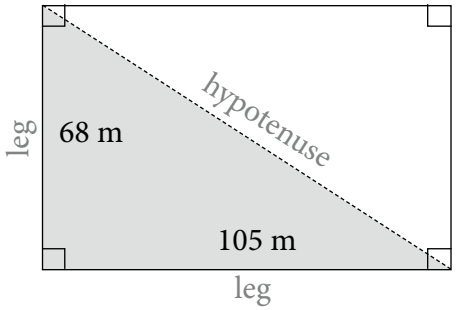
Let's take a look at a real-life example of how the Pythagorean theorem can help us find a missing distance.

Example: Suppose you know that a certain soccer field[1] has a length of 105 m and a width of 68 m. If we're standing at one corner, how far would we walk if we walked diagonally across the field?



Notice that since a soccer field is a rectangle (and a rectangle has all right angles), we can think of the diagonal as dividing the field into two

identical right triangles. We know the legs of the right triangles, but we need to find the hypotenuse (the diagonal). We can use the Pythagorean theorem to find this!



Pythagorean theorem:

$a^2 + b^2 = c^2$

Substitute known lengths — we know the lengths of the two legs (a and b — it doesn't matter which leg we view as a and which one we view as b):

$\mathbf{(68\ m)}^2 + \mathbf{(105\ m)}^2 = c^2$

Simplify:

$\mathbf{4{,}624\ m^2 + 11{,}025\ m^2} = c^2$

$\mathbf{15{,}649\ m^2} = c^2$

Now we've found c^2, but we still need to find c. We can easily do this by taking the square root of both sides of the equation.

$\sqrt{15{,}649\ \text{m}^2} = \sqrt{c^2}$

While finding the square root of 15,649 m^2 would be rather tedious manually, we can find it easily on a calculator, giving us the answer: a number that rounds to 125.1 m.

$\mathbf{125.1\ m} = \boldsymbol{c}$

Keeping Perspective

Just as we can take the square root of the area to find the length of each side of a square, we can use square roots to help us find missing sides in a right triangle. In both cases, we're using the information we know to find the information we don't know.

As you learn more about exploring the complexities of God's creation, you'll find that the relationships will gradually become more involved and require more concepts to solve. Don't let the complexities overwhelm you — just remember to break each problem into steps. Above all, remember that the very complexities point us to God's greatness — He created and sustains each relationship! Truly His wisdom and might know no bounds.

17.6 More with the Pythagorean Theorem

In working with exponents and unknowns, sometimes we'll encounter equations that require more than one step to solve. Don't let multi-step problems intimidate you: break them down into steps.

For example, consider this problem:

$$x^2 + 16 = 80$$

Here, x^2 isn't on a side by itself to start with, but there's no need to panic. We know that we can add the *same* amount to *both* sides of the equation. If we add –16 to both sides, then, we'll be left with x^2 on a side by itself and can then find x by taking the square root.

Example: $x^2 + 16 = 80$

Add –16 to both sides of the equation:

$x^2 + 16 + \mathbf{-16} + = 80 + \mathbf{-16}$

$x^2 = \mathbf{64}$

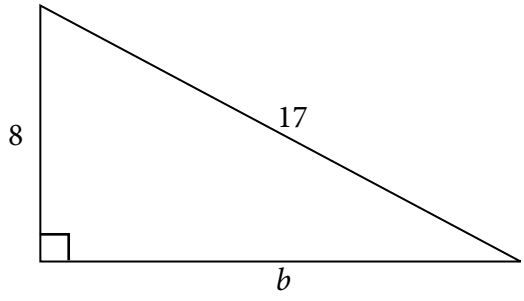
Take the square root of both sides:

$\sqrt{x^2} = \sqrt{64}$

$\boldsymbol{x = 8}$

In the last lesson, we only used the Pythagorean theorem to find the hypotenuse — the side we've been representing with *c*. But with a few more steps, we can use the Pythagorean theorem to find *any* missing side in a right triangle.

Example: Find the missing side in this right triangle.



Pythagorean theorem:

$a^2 + b^2 = c^2$

Substitute known values:

$\mathbf{8^2} + b^2 = \mathbf{17^2}$

Simplify exponents:

$\mathbf{64} + b^2 = \mathbf{289}$

Add –64 to *both sides* of the equation:

$64 + \mathbf{-64} + b^2 = 289 + \mathbf{-64}$

Simplify:

$b^2 = \mathbf{225}$

Find the square root of *both sides* of the equation:

$\sqrt{b^2} = \sqrt{225}$

$\boldsymbol{b = 15}$

Keeping Perspective

Much as some works of art require only a couple of techniques, while others require many (combining colors, special brush strokes, painting in layers, etc.), so some math problems require only a couple of techniques to solve, while others require many. As you practice combining different techniques, remember that God's the artist of the complex picture math attempts to describe. While men have developed techniques, God is the One who gives those techniques meaning by holding all things together consistently. He's the One who makes working with unknowns possible.

17.7 Converse of the Pythagorean Theorem

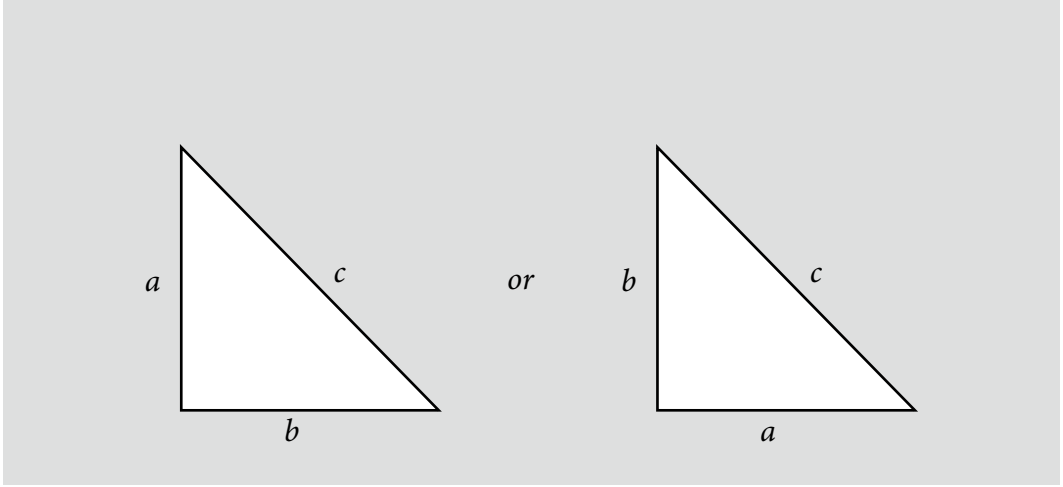
Not only can the Pythagorean theorem help us find missing dimensions in a right triangle, but it can also help us tell if a triangle is indeed a right triangle. The Pythagorean theorem describes the fact that *if a triangle is a right triangle, then the sum of the squares of the lengths of the legs will equal the square of the length of the hypotenuse*; the opposite argument, or **converse**, of that is that *if the sum of the squares of the lengths of the legs in a triangle equal the square of the length of the hypotenuse, the triangle is a right triangle*.

While not all converse, or opposite, arguments hold true (just because every toddler cries does not mean that everyone who cries is a toddler), in the case of the Pythagorean theorem, the converse holds true. If the sum of the squares of the lengths of the legs in a triangle equal the square of the length of the hypotenuse, the triangle has to be a right triangle. We call this the **converse of the Pythagorean theorem**.

Converse of the Pythagorean Theorem

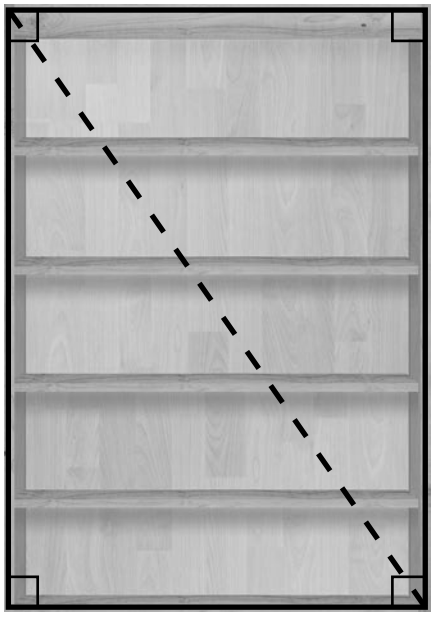
If the sum of the squares of the lengths of the legs in a triangle equal the square of the length of the hypotenuse, the triangle is a right triangle.

In other words, if in a triangle, $a^2 + b^2 = c^2$, then the triangle is a right triangle.



Why do we care that the converse of the Pythagorean theorem is true? Well, let's say we want to make sure we've assembled a sturdy bookcase. The sides of the bookcase should be nailed to the base at right (90°) angles — they can't intersect at a different angle, or the bookcase would totter (not a good thing!). How can we tell if we've got the shelves nailed at the correct angle?

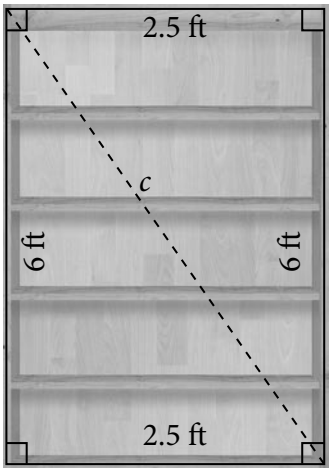
One way is to use the converse of the Pythagorean theorem. A properly built bookshelf would be a rectangle, with all right angles. If the bookcase is truly a rectangle, then a diagonal line from corner to corner would divide the bookcase into two right triangles.



A Properly Built Bookcase

(All right angles; a diagonal line from corner to corner would divide the bookcase into two right triangles.)

Now we can measure the dimensions of a bookcase we're building. We could then use the Pythagorean theorem to figure out what the length of the diagonal from corner to corner would be if the diagonal does indeed form two right triangles. If the actual measurement of the diagonal equals the measurement calculated using the Pythagorean theorem, then we know that the bookcase forms two right triangles, thus telling us that its angles are indeed right angles. If it's different, though, we'd know that we need to adjust the bookcase some more.



Example: If a rectangular bookcase is 6 ft tall and 2.5 ft wide, what would the length marked *c* in the picture have to measure in order for the bookcase to be built with right angles?

Substitute known values:

$a^2 + b^2 = c^2$

$\mathbf{(6\ ft)}^2 + \mathbf{(2.5\ ft)}^2 = c^2$

Simplify exponents:

$\mathbf{36\ ft^2 + 6.25\ ft^2} = c^2$

Simplify:

$\mathbf{42.25\ ft^2} = c^2$

Take the square root of *both sides*:

$\sqrt{42.25\ \text{ft}^2} = \sqrt{c^2}$

$\mathbf{6.5\ ft} = c$

So *if* the diagonal (c) measures 6.5 ft, then the triangles must be right triangles, meaning the bookcase is built with right angles. If it does not, then we'd better keep tweaking the bookcase until it does.

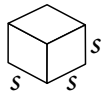
Keeping Perspective

Since many shapes can be thought of as several triangles, triangles can help us find measurements we might not expect! As we saw in Book 1, shapes help us describe all sorts of aspects of God's creation. By combining what we know about shapes with the techniques we've learned for finding unknowns, we're able to solve even more real-life problems. No matter how many techniques we combine, though, we're ultimately relying on the underlying consistencies God created and sustains and on the fact that God created us capable of appreciating and recording those consistencies.

17.8 Other Roots

Up until now, the only roots we've looked at have been square roots. As we've seen, a square root is a way of asking for the number that, times itself, equals a given number. But let's say that we wanted to find a number that, times itself *three* times (rather than two), equaled a given number. For example, say we knew the volume of a cube was 27 in^3, but wanted to find out the measure of each side of the cube. We want to find a number that, times itself three times, equals 27 in^3. Is there a way to do that?

$Volume = s^3$



Yes! All we'd need to do is put a superscript 3 in front of the root symbol. This would tell us to find the *cubed* root — the number that, times itself *three* times, equals a given number.

$\sqrt[3]{27}$ *asks for the number that, times itself three times, equals 27.*

Answer is 3, as 3 • 3 • 3 equals 27.

Whatever superscript number we put to the left of the root sign tells us what root were looking for — that is, the number of times the root would be multiplied by itself to equal the number under the root sign (called the **radicand**).

Root	Meaning	Result
$\sqrt{4}$ *or* $\sqrt[2]{4}$	*the number that, times itself, equals 4*	2, as 2 • 2 = 4
$\sqrt[3]{8}$	*the number that, times itself three times, equals 8*	2, as 2 • 2 • 2 = 8
$\sqrt[4]{16}$	*the number that, times itself four times, equals 16*	2, as 2 • 2 • 2 • 2 = 16
$\sqrt[5]{32}$	*the number that, times itself five times, equals 32*	2, as 2 • 2 • 2 • 2 • 2 = 32
$\sqrt[6]{64}$	*the number that, times itself six times, equals 64*	2, as 2 • 2 • 2 • 2 • 2 • 2 = 64

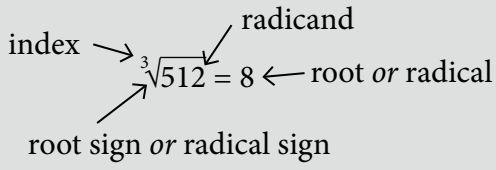
…etc.

Notice from the chart that we *could* put a 2 next to the root sign to show the squared root, only it's customary not to bother.

In other words, just as exponents can be used to show repeated multiplication, **roots help us express repeated division**. Just as we can use different exponents to show how many times we want to multiply a number, we can use different roots to reverse that repeated multiplication.

Term Time

Radical is another name for a root, **index** is the name for the superscript number that indicates which root we're taking, **radical sign** is another name for the root sign, and **radicand** is a term used to refer to "the quantity under a radical sign."[2]



It's important to note that the index is not normally written when we're finding the square root. If there's no index, the radical sign by itself means the square root. For example, $\sqrt{16}$ means the same thing as $\sqrt[2]{16}$.

To avoid confusion with an index (the superscript number that indicates which root we're taking), when writing a multiplication next to a square root sign, use parentheses. For example, even though $3\sqrt{64}$ technically means $3 \cdot \sqrt{64}$, if handwritten, it could be easy to mistake the 3 for an index (as in $\sqrt[3]{64}$). So it's safer to develop the habit of including parentheses. For example, write $3(\sqrt{64})$ or $(3)\sqrt{64}$ rather than $3\sqrt{64}$.

Finding Various Roots

It was easy to find the cubed root of 27, since it was a relatively small number. Often, we'll need to factor to find various roots. In this course, **we'll focus on square roots and cubed roots, but the same process applies for finding other roots.**

Example: Find the cubed root of 125 (also written $\sqrt[3]{125}$).

Here we want to find the number that, times itself three times, equals 125. We can figure this out by factoring 125.

$$\begin{array}{l} 125 \\ \ \ \wedge \\ 5 \cdot 25 \\ \ \ \ \ \wedge \\ \ \ 5 \cdot 5 \end{array}$$

Since $5 \cdot 5 \cdot 5$ equals 125, 5 is the cubed root of 125.

Example: Find the cubed root of 216 (also written $\sqrt[3]{216}$).

Here we want to find the number that, times itself three times, equals 216. We can figure this out by factoring 216.

$$\begin{array}{l} 216 \\ \ \ \wedge \\ 3 \cdot 72 \\ \ \ \ \ \wedge \\ \ \ 8 \ \ \cdot \ \ 9 \\ \ \ \wedge \ \ \ \ \wedge \\ 4 \cdot 2 \ \ 3 \cdot 3 \\ \wedge \\ 2 \cdot 2 \end{array}$$

Notice that none of the prime factors multiplied together three times equal 216. Thus, we have to look at how we could *combine* the prime factors to find a cubed root. Let's arrange all the prime factors in order and look at them: $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

Notice that we have three 2s and three 3s. We can combine these to find the factor that, when multiplied by itself 3 times, equals 216.

$$\begin{array}{l} 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 216 \\ \ \ \ \ \ 6 \cdot 6 \cdot 6 \ \ \ \ \ \ = 216 \end{array}$$

The cubed root of 216 is 6, as $6 \cdot 6 \cdot 6$ equals 216.

There's also a way to find various roots on a calculator using a button that typically has this symbol on it: $\sqrt[x]{y}$. However, for now, you'll need to do your own calculating of cubed roots. While the process may seem tedious, finding a root via factoring will help you familiarize yourself with roots and with the factors that make up numbers, which will help you in understanding many different aspects of math.

Keeping Perspective

Roots, or **radicals**, are another helpful notation to aid us in describing God's creation. As with all the other notations we've looked at, they're part of the "language" of mathematics that we use to concisely describe and work with real-life quantities and consistencies. While much of math focuses on the notations, we need to remember the purpose of learning those notations: to better understand God's creation and to be equipped to use math to His glory in whatever situations He brings into our lives.

17.9 Chapter Synopsis and the Pythagorean Worldview

Well, we've reached the end of our short look at square roots and unknowns. We started by looking at **square roots** and how they undo squaring a number. We then applied that knowledge to help us find various unknowns.

We also explored in depth the relationship between the legs and hypotenuse in a right triangle: the **Pythagorean theorem** ($a^2 + b^2 = c^2$). We used this theorem to find the length of any side of a right triangle when we knew the other two sides, and we used the **converse of the Pythagorean theorem** to tell if a triangle was really a right triangle. Last, we touched on other roots besides square roots.

In ending our explorations, it's fitting that we take a look at the mathematician for which the Pythagorean theorem is named . . . and at how his very theorem contradicted his worldview.

Pythagoras and the Pythagorean Worldview[3]

The Pythagorean theorem is named after a Greek mathematician named Pythagoras. Pythagoras was one of the earlier Greek mathematicians, born somewhere around 570 B.C. He didn't always live in Greece — he spent some time in Egypt, where he may have learned some of their mathematical and religious teachings.

Pythagoras is known, however, for the combination school/religious brotherhood that he founded in Greece. Pythagoras and those in his "brotherhood" are collectively referred to as Pythagoreans.[4] While the Pythagoreans contributed a lot to math, they also worshiped numbers.

> *In fact, he and his followers worshiped numbers as the full intelligibility and the generating source of all things. This veneration is revealed in one of the Pythagorean creedal confessions, "Bless us divine number, thou who generatest gods and men."*
> *— James Nickel,* Mathematics: Is God Silent?[5]

Pythagoras
(c. 570 B.C.–490 B.C.)

The Pythagorean . . . having been brought up in the study of mathematics, thought that things are numbers . . . and that the whole cosmos is a scale and a number. — Aristotle

The Pythagoreans were very secretive. Those members on the "inside," so to speak, were "bound by oath not to reveal the teaching or secrets of the school."[6] Thus, history is not always totally clear about some of the school's discoveries. We don't even know which discoveries Pythagoras made versus others at the school, since it was common to give Pythagoras credit for everything.

All throughout history, men have tried to explain math and the order in creation apart from acknowledging God. Their theories were believed for a time and disproven later. This proved true for the Pythagorean belief too. In fact, as James Nickel points out in *Mathematics: Is God Silent?* the Pythagorean theorem *itself* disproves the Pythagorean worldview![7]

You see, one of the strange teachings of the Pythagorean school was that they believed everything (including fractions) could be reduced one way or another to whole numbers (1, 2, 3, . . .). Even numbers we wouldn't think of as whole (such as $\frac{8}{9}$) can be thought of as a whole number divided by a whole number. In other words, they thought of everything in terms of positive rational numbers.

Yet when we apply the Pythagorean theorem, we encounter numbers that *can't* be described using whole numbers alone.

Consider a right triangle whose legs are both 1.

Pythagorean theorem:

$$a^2 + b^2 = c^2$$

Substitute known values:

$$\mathbf{1}^2 + \mathbf{1}^2 = c^2$$

Simplify:

$$\mathbf{1 + 1} = c^2$$

$$\mathbf{2} = c^2$$

Find the Square Root of Both Sides:

$$\sqrt{2} = \sqrt{c^2}$$

$$\sqrt{2} = \boldsymbol{c}$$

The hypotenuse of a triangle whose legs are both 1 cannot be expressed using rational numbers. It is the $\sqrt{2}$, an irrational number that begins 1.414213562 . . . and continues on and on. There's no fraction or ratio made up of whole numbers that can express this quantity precisely. Not everything can be reduced to a positive rational number! God's creation is much more complicated than that.

As we look at the Pythagorean school and their thinking, and then see how their very theorem contradicted their worldview, we're reminded of the foolishness of man's sinful thinking. God's truth is written in His Word for any who will listen, yet all throughout history, men have tried to ascribe God's glory elsewhere. While most modern mathematicians don't boldly worship numbers the same way the Pythagoreans did, that same spirit of enthroning math and the creation rather than the Creator is still something we need to guard against, both in what we read and in our own lives.

For the wisdom of this world is foolishness with God. For it is written, He taketh the wise in their own craftiness (1 Corinthians 3:19).

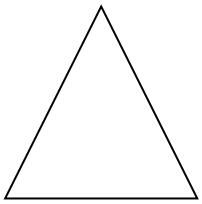
Thy word is true from the beginning: and every one of thy righteous judgments endureth for ever (Psalm 119:160).

Because that which may be known of God is manifest in them; for God hath shewed it unto them. For the invisible things of him from the creation of the world are clearly seen, being understood by the things that are made, even his eternal power and Godhead; so that they are without excuse: Because that, when they knew God, they glorified him not as God, neither were thankful; but became vain in their imaginations, and their foolish heart was darkened. Professing themselves to be wise, they became fools, And changed the glory of the uncorruptible God into an image made like to corruptible man, and to birds, and fourfooted beasts, and creeping things. Wherefore God also gave them up to uncleanness through the lusts of their own hearts, to dishonour their own bodies between themselves: Who changed the truth of God into a lie, and worshipped and served the creature more than the Creator, who is blessed for ever. Amen (Romans 1:19–25).

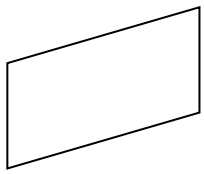
Appendix A: Reference

Polygon

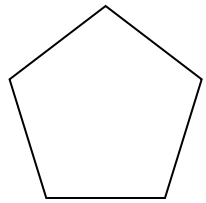
(closed, two-dimensional figure with straight lines)



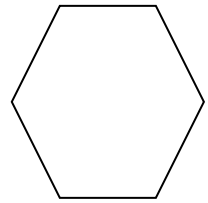
Triangle
(3 sides)



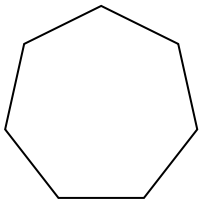
Quadrilateral
(4 sides)



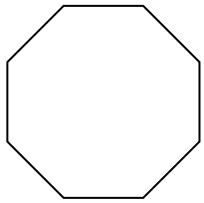
Pentagon
(5 sides)



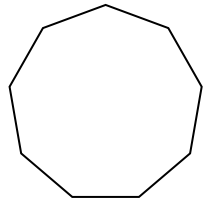
Hexagon
(6 sides)



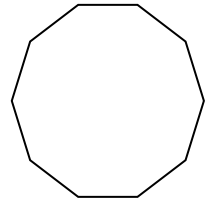
Heptagon
(7 sides)



Octagon
(8 sides)



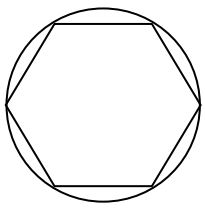
Nonagon
(9 sides)

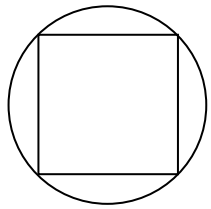


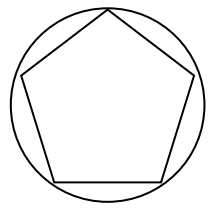
Decagon
(10 sides)

Regular Polygon

(All sides are equal; all edges would touch a circle drawn around the figure, as all angles are the same.)

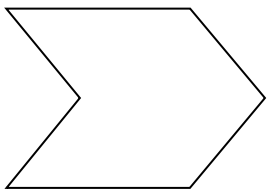


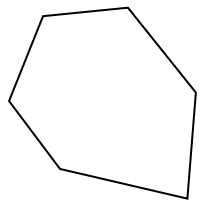


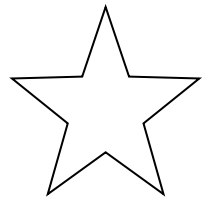


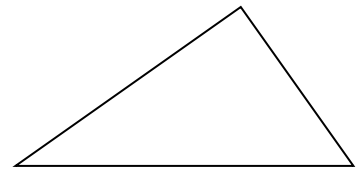
Irregular Polygon

(Polygons that are not regular.)





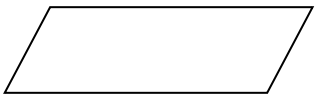




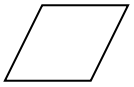
Specific Quadrilaterals

Trapezoid
(quadrilateral with 1 pair of parallel sides)

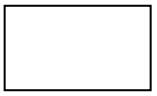
Some books define a trapezoid as a quadrilateral with 1, and only 1, pair of parallel sides, while others as a quadrilateral with 1 (or more) pair of parallel sides. Likewise, some define a rhombus/diamond differently than listed here. Always remember that definitions can — and do — vary!



Parallelogram
(quadrilateral with both pairs of opposite sides parallel)



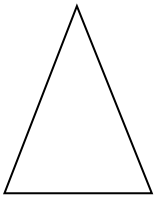
Rhombus/Diamond
(parallelogram with equal-length sides)



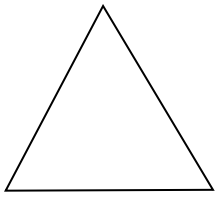
Rectangle
(parallelogram with right angles)

Square
(parallelogram with equal-length sides *and* right angles)

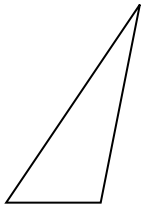
Triangles Categorized by Length of Sides



Isosceles
(two equal sides)

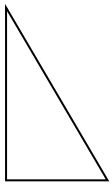


Equilateral
(all equal sides)

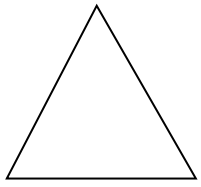


Scalene
(no equal sides)

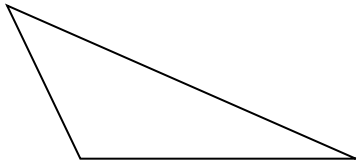
Triangles Categorized by Angles



Right Triangle
(a right angle)



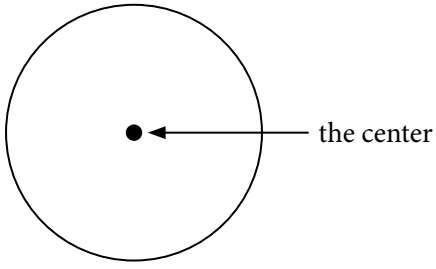
Acute Triangle
(all acute angles)



Obtuse Triangle
(an obtuse angle)

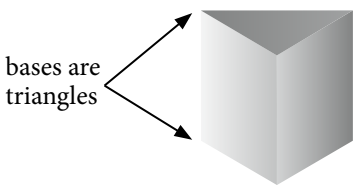
Circle

(closed two-dimensional figure; each part of the edge is equally distant from the center)

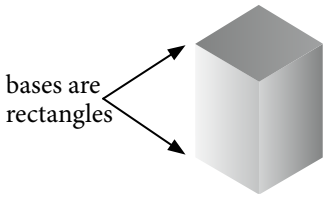


Prism

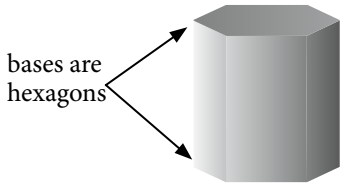
(A solid with two bases that are parallel polygons, and faces [sides] that are parallelograms; the prism is named after the shape of the bases.)



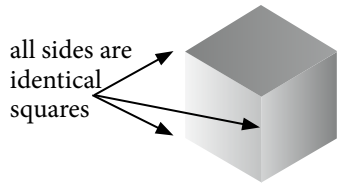
Triangular Prism



Rectangular Prism



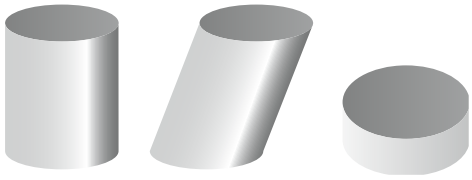
Hexagonal Prism



Cube

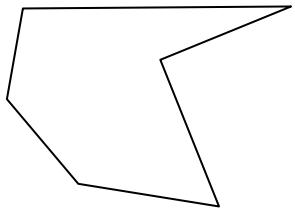
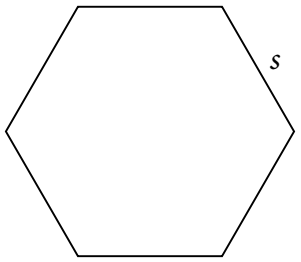
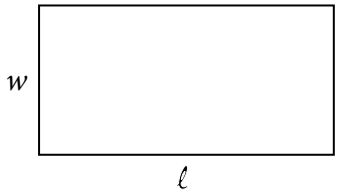
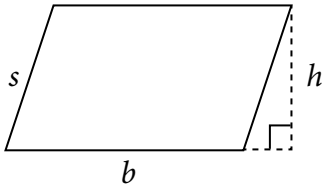
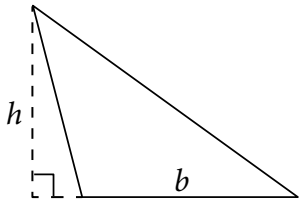
Cylinder

(A solid with two bases that are equal parallel circles, having an equal diameter in any parallel plane between them.)

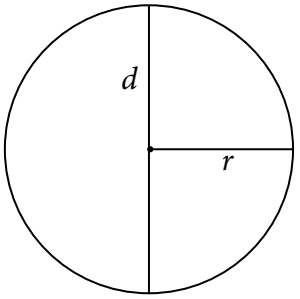


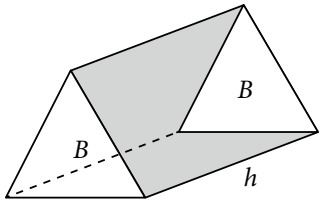
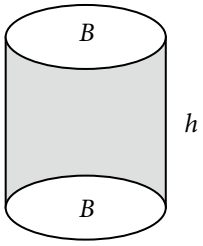
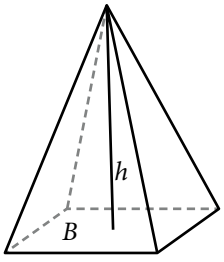
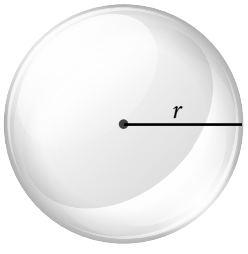
Prism and cylinder definitions were based on *Ray's New Higher Arithmetic*, Revised (Cincinnati: Van Antwerp, Bragg & Co., 1880), p. 390.

Formulas

Shape Name	Type of Shape	Perimeter	Area
Polygons		***P* = *sum of the lengths of each side*** *or* $P = s_1 + s_2 \ldots s_n$	View as multiple triangles or other shapes.
Regular Polygon		***P* = *(number of sides)* • *(length of a side)*** *or* $P = n \cdot s$ *or* $P = n(s)$ *or* $P = ns$	View as multiple triangles or other shapes.
Rectangle		$P = (2 \cdot length) + (2 \cdot width)$ *or* $P = 2 \cdot \ell + 2 \cdot w$ *or* $P = 2(\ell) + 2(w)$ *or* $P = 2\ell + 2w$	$A = length \cdot width$ *or* $A = \ell \cdot w$ *or* $A = \ell(w)$ *or* $A = \ell w$
Square		$P = 4 \cdot side$ *or* $P = 4 \cdot s$ *or* $P = 4(s)$ *or* $P = 4s$	$A = side \cdot side$ *or* $A = s \cdot s$ *or* $A = s(s)$ *or* $A = s^2$
Parallelogram		$P = (2 \cdot base) + (2 \cdot side)$ *or* $P = 2 \cdot b + 2 \cdot s$ *or* $P = 2(b) + 2(s)$ *or* $P = 2b + 2s$	$A = base \cdot height$ *or* $A = b \cdot h$ *or* $A = b(h)$ *or* $A = bh$
Triangle		***Perimeter* = *sum of the lengths of each side*** *or* $P = s_1 + s_2 + s_3$	$A = \frac{1}{2} \cdot base \cdot height$ *or* $A = \frac{base \cdot height}{2}$ $A = \frac{1}{2} \cdot b \cdot h$ *or* $A = \frac{1}{2}bh$ *or* $A = \frac{b \cdot h}{2}$ *or* $A = \frac{bh}{2}$

A = Area
B = area of the base
b = base
C = circumference
d = diameter
h = height
ℓ = length
n = number of sides
P = Perimeter
π = 3.14
r = radius
s = side
V = Volume
w = width (or height)

Shape Name	Type of Shape	Circumference	Area
Circle		***Circumference* = π • *diameter*** *or* $C = \pi \cdot d$ *or* $C = \pi(d)$ *or* $C = \pi d$ ***Circumference* = 2 • π • *radius*** *or* $C = 2 \cdot \pi \cdot r$ *or* $C = 2(\pi)(r)$ *or* $C = 2\pi r$	***Area* = π • *radius* • *radius*** *or* $A = \pi \cdot r^2$ *or* $A = \pi(r^2)$ *or* $A = \pi r^2$
		diameter* = 2 • *radius *or* $d = 2 \cdot r$ *or* $d = 2r$ ***radius*** $= \frac{1}{2} \cdot$ ***diameter*** *or* $r = \frac{1}{2} \cdot d$ *or* $r = \frac{1}{2} d$	

Shape Name	Type of Shape	Volume	Area
Prism		***Volume* = *area of base* • *height*** *or* $V = B \cdot h$ *or* $V = Bh$	***Total surface area* = *area of all the sides (i.e., surfaces) of a solid object***
Cylinder		***Volume* = *area of base* • *height*** *or* $V = B \cdot h$ *or* $V = Bh$	did not cover
Pyramid		***Volume*** $= \frac{1}{3} \cdot$ ***area of base* • *height*** *or* $V = \frac{1}{3} \cdot B \cdot h$ *or* $V = \frac{1}{3} Bh$	did not cover
Sphere		***Volume*** $= \frac{4}{3} \cdot \pi \cdot$ ***radius***^{3} ***or*** $V = \frac{4}{3} \cdot \pi \cdot r^3$ ***or*** $V = \frac{4}{3} \pi r^3$	did not cover

A = Area
B = area of the base
b = base
C = circumference
d = diameter
h = height
ℓ = length
n = number of sides
P = Perimeter
π = 3.14
r = radius
s = side
V = Volume
w = width (or height)

Units of Measure

Distance

Distance – U.S. Customary

12 inches (in) = 1 foot (ft)
3 feet / 36 inches = 1 yard (yd)
1,760 yard / 5,280 feet = 1 mile (mi)

Conversion Between Systems

1 inches (in) = 2.54 centimeter (cm)
1 foot (ft) = 30.48 centimeter (cm)
1 yard (yd) = 0.9144 meter (m)
1 mile (mi) = 1.609344 kilometer (km)

Distance – Metric/SI

10 millimeters (mm) = 1 centimeter (cm)
10 centimeters = 1 decimeter (dm)
10 decimeters / 100 centimeters / 1,000 millimeters = 1 meter (m)
10 meters = 1 decameters (dam)
10 decameters = 1 hectometer (hm)
10 hectometers / 1,000 meters = 1 kilometer (km)

Time

60 seconds (s) = 1 minute (min)
60 minutes = 1 hour (hr)
24 hours = 1 day (d)
7 days = 1 week (wk)
365 days = 1 year (yr *or* y)

Liquid Capacity

U.S. Customary

3 teaspoons (tsp) = 1 tablespoon (Tbsp)
16 tablespoons = 1 cup (c)
2 cups = 1 pint (pt)
2 pints = 1 quart (qt)
4 quarts = 1 gallon (gal)

2 tablespoons (Tbsp) ≈ 1 fluid ounce (fl oz)
8 fl oz = 1 cup (c)
16 fl oz = 1 pint (pt)
32 fl oz = 1 quart (qt)
128 fl oz = 1 gallon (gal)

Conversion Between Systems

1 teaspoon ≈ 5 milliliters
1 gallon = 3.78541 liters

1 pint = 28.875 in^3
1 quart = 57.75 in^3
1 gallon = 231 in^3

Metric

10 milliliters (ml *or* mL) = 1 centiliter (cl *or* cL)
10 centiliter / 100 milliliter = 1 deciliter (dl *or* dL)
10 deciliter / 100 centiliter / 1,000 milliliter = 1 liter (*l or* L)
10 liters = 1 dekaliter (dal *or* daL)
10 dekaliter = 1 hectoliter (hl *or* hL)
10 hectoliter / 1,000 liters = 1 kiloliter (kl *or* kL)

Dry Capacity

U.S. Customary

2 pints (pt) = 1 quart (qt)
8 quart = 1 peck (pk)
4 peck = 1 bushel (bu) / 32 quarts (qt)

Conversion Between Systems

1 quart = 67.2006 inches^3
1 bushel = 2,150.42 inches^3

Note: The pint and quart here represent a larger capacity than the ones measuring liquid—they should not be used interchangeably. Unless the problem specifically states otherwise, you can assume pint and quart in this course refer to the liquid units.

Mass

U.S. Customary

16 ounces (oz) = 1 pound (lb)
2,000 pounds = 1 ton (called a "short ton")

Conversion Between Systems

1 ounce = 28.3495 grams
1 pound = 453.592 grams
1 U.S. ton (called a short ton) = 0.907185 metric tons

Note: These ounces are different than the fluid ounces listed under liquid capacity.

Metric

10 milligrams (mg) = 1 centigram (cg)
10 centigrams / 100 milligrams = 1 decigram (dg)
10 decigrams / 100 centigrams / 1,000 milligrams = 1 gram (g)
10 grams = 1 dekagram (dag)
10 dekagrams = 1 hectogram (hg)
10 hectograms / 1,000 grams = 1 kilogram (kg)

For more unit details, see the official standards given in Tina Butcher, Linda Crown, Rick Harshman, and Juana Williams, eds. *NIST Handbook 44: 97th National Conference on Weights and Measures 2012*, 2013 ed. (Washington: U. S. Department of Commerce, 2012), Appendix C. Found on http://www.nist.gov/pml/wmd/pubs/h44-13.cfm, accessed 10/6/2014.

Important Mathematical Relationships

Distance Problems *See Lessons 3.5, 3.7, and 5.3.*

$$d = s \cdot t \text{ or } d = st$$

distance = speed • time

Altitude (or Height) Problems *See Lesson 3.7.*

$$a = r \cdot t + a_i$$

altitude (or height) = rate of change in altitude (or height) • time + initial altitude (or height)

Percent Problems *See Lesson 6.3.*

$$R \cdot B = P \text{ or } RB = P$$

Rate • Base = Percentage

Think: The rate *of* the base *equals* the percentage.

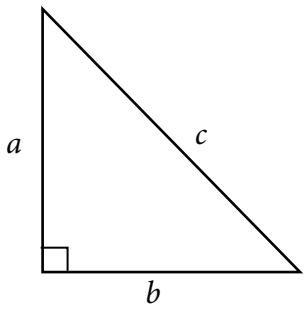
Probability *See Chapter 12.*

$$\textit{probability of an event} = \frac{\textit{outcomes that produce event}}{\textit{total possible outcomes}}$$

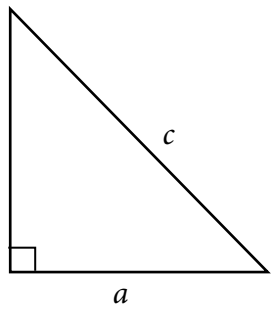
$$\textit{odds for} = \frac{\textit{outcomes that produce event}}{\textit{outcomes that do not produce event}}$$

$$\textit{odds against} = \frac{\textit{outcomes that do not produce event}}{\textit{outcomes that produce event}}$$

Pythagorean Theorem *See Lessons 17.5–17.7.*



or



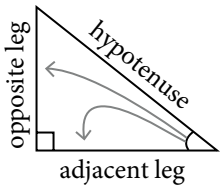
$a^2 + b^2 = c^2$

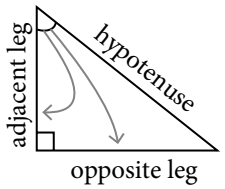
where c stands for the length of the hypotenuse of a right triangle, and a and b stand for the length of the two legs of the same right triangle

Trigonometry *See Chapter 18.*

Tangent of an angle — The ratio between the opposite and adjacent leg of either non-right angle in a right triangle.

$$\textit{tangent of an angle} = \frac{\textit{opposite leg}}{\textit{adjacent leg}}$$





Sine of an angle — The ratio between the opposite leg of either non-right angle in a right triangle and the hypotenuse.

$$\textit{sine of an angle} = \frac{\textit{opposite leg}}{\textit{hypotenuse}}$$

Cosine of an angle — The ratio between the adjacent leg of either non-right angle in a right triangle and the hypotenuse.

$$\textit{cosine of an angle} = \frac{\textit{adjacent leg}}{\textit{hypotenuse}}$$

Exponential Growth and Decay (Applies to Compound Interest Too) *See Lessons 15.5–15.7 and Lessons 20.1–20.2.*

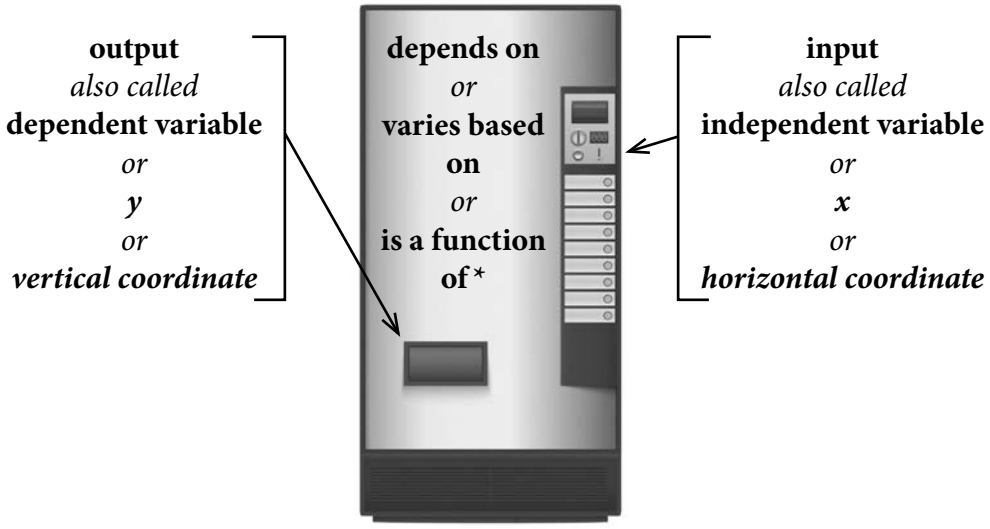
$$P = P_0(1 + r)^t$$

where

P = final amount after growth or decay
P_0 = initial amount before growth or decay
r = rate of growth or decay over a specified period of time
t = the number of specified periods of time that have passed

Mathematical Relationships See Lesson 13.2–13.5 and 14.1 for an explanation.

Important Terms



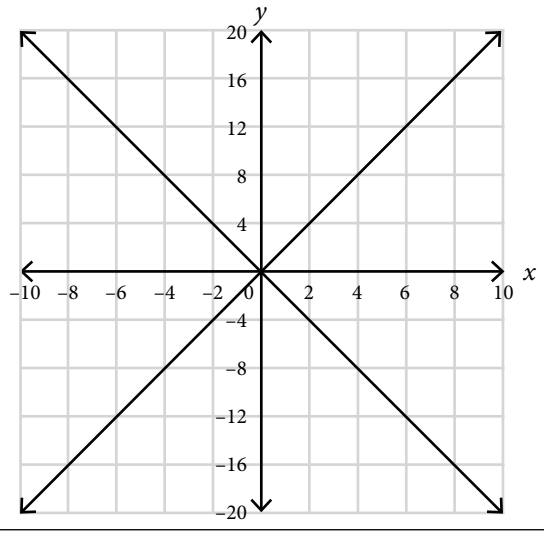
*The term *is a function of* is only used if we know there's only one output for every input.

Appendix B: Mathematical Relationships: Graphs and Functions

It's very easy to tell on a graph if the relationship between an input and output is a function or not. Remember, a function means that for every input there's one output. When you look at a graph, you can easily see if for every *x* value (i.e., every input), there's only one *y* value (i.e., one output). How? Well, using your finger, trace the *x*-axis horizontally. At any point does the line(s) or curve have two different vertical values? If it does, then it is not a function, as not every input has only one output.

Not a Function

(As we travel across the graph horizontally, there is *not* only one vertical value for *every* horizontal value. Thus, *each* input does not only have *one* output.)



Function

(As we travel across the graph horizontally, there's only one vertical value for every horizontal position along the line or curve. There is only one output for each input.)